Renormalon for threshold-like asymptotics of space-like parton correlators

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Outline

- Introduction to marginal asymptotics in local QFT.
- Threshold asymptotics of quark-bilinear coefficient function
- Lesson from 2D large N Gross Neveu

UV/IR fixed-point and local QFT

- Massive scaling-limits for near-critical lattice model

1. $\langle \sigma(x\zeta)\sigma(0)\rangle|_{\zeta\to\infty} \to Z(\zeta)W(x)$. Massive scaling function $W(x)$.
2. $W(x) \to e^{-|x|}, |x| \to \infty;$ $W(x) \to \frac{1}{|x|^{2d}} \ln^{\frac{\gamma_1}{\beta_0}}|x|, |x| \to 0.$
3. Schwinger functions (Euclidean-time Green functions) for a local-QFT.
4. Universal short-distance asymptotics controlled by UV CFT.

- For critical lattice system with $\zeta = \infty$, one has universal massless scaling limits $\langle \sigma(x)\sigma(0)\rangle|_{x\to\infty} \to \frac{A}{|x|^{2d}} \ln^{\frac{\gamma_1}{\beta_0}}|x|$ controlled by IR CFT.
- UV of IR (massive local QFT) = IR of UV (critical lattice model).

A text-book example of sub-critical theory

- 2D Ising CFT/Massive Ising QFTs. $h = 0, T \to T_c^{\pm}$ massive scaling functions worked out in mid 1970s by Barry McCoy.

1. $W(x)|_{x\to 0} \to \frac{1}{x^{\frac{1}{4}}}\left(1 + \frac{x}{2}\ln\frac{e^{\gamma_E}x}{8} + \frac{x^2}{16} + \frac{x^3}{32}\ln\frac{e^{\gamma_E}x}{8} + \cdots\right)$
2. Asymptotic description in terms of perturbation theory to UV CFT: $\langle\sigma(x)\sigma(0)\exp\int d^2x\, m\varepsilon(x)\rangle_{Ising\ CFT}$
3. The perturbation is relevant(sub-critical) → finite number of logarithms at each power. Absence of logarithm at leading power.
4. Perturbative expansion = power expansion. Fast decoupling of UV/IR. Borel summable if no vacuum degeneration.

Marginal theory

- What if the dimension of the perturbation = space-time dimension ?

  Then the perturbation is called marginal. Infinitely many logarithms.

- Marginal perturbative series is tricky to handle and has certain special features.

1. Fixed order PT suffers UV/IR divergences and must be renormalized, resulting in increasing numbers of logarithms that might destroy the asymptotic nature of the PT expansion.
2. Can be handled only after the discovery of RGE structure in early 1970s. Key observation: logarithms re-summed into running couplings.
3. Marginally relevant (asymptotically safe in UV) and Marginally irrelevant (AS in IR).

Marginal theory: examples

1. QCD. UV CFT = free quarks and gluons. Perturbation theory = QCD perturbation theory.
2. Massless 4D scalar $\lambda\phi^4$. IR limit of 4D critical Ising. IR CFT = free scalar (4D scalar "triviality". Proven in 2019.) 4D massless QED is similar.
3. O(N) non-linear sigma models && Gross-Neveu models in 2D. On-shell integrability. UV CFTs = free "pions" or fermions. QCD like.
4. $\beta^2 \to 8\pi$ Sine Gordon && $g\bar{J}^a J^a$ perturbation to $c = 1$ SU(2) CFT in 2D. UV or IR depending on the sign of g. Generalizable to $c = \frac{3k}{k+2}$ SU(2) CFT ($su(2)_k$ WZW models). IR limit of critical AF integrable spin chains with $s = \frac{k}{2}$.
5. In special cases, marginal perturbation to a CFT is another CFT: $N = 4$ maximally supersymmetric CFT in 4D && "ϵ expansions".

Marginal PT: A simple example

- The Prime Number Theorem:

$$\pi(x) \to \frac{x}{\ln x}\sum_{n=0}^{\infty}\frac{n!}{\ln^n x}, x \to \infty.$$

1. $\alpha(x) = \frac{1}{\ln x}$: the "running coupling constant".
2. Non-alternating factorial growth: Borel non-summable.
3. Borel re-summation: $\int_0^x \frac{dt}{\ln t}$ has a singularity at $t = 1$. "Renormalon-singularity".
4. Large power-corrections: $\pi(x) - PV \int_0^x \frac{dt}{\ln t} = O(\sqrt{x})$. "Resurgence" fails.

General Features of Marginal PT (UV limit, IR is similar)

- Euclidean correlator: $G(zm) = z^{2d_O}\langle 0|O(z)O(0)|0\rangle$. Universal $z^2 \to 0$ asymptotic expansion:

$$G(zm) \to \alpha(z)^{\frac{-2\gamma_O^1}{\beta_0}} \exp \int_0^{\alpha(z)} \left(\frac{2\gamma_O^1}{\beta_0 \alpha} - \frac{2\gamma_O(\alpha)}{\beta(\alpha)}\right) d\alpha \, G_{PT}[\alpha(z)] + O(mz).$$

1. $\alpha(z) \to \frac{1}{\beta^0 \ln(\frac{1}{\mathrm{mz}})}$: the running coupling constant.
2. Minimal scheme: $\frac{1}{\alpha} + \frac{\beta_1}{\beta_0} \ln \alpha = \beta_0 \ln \frac{1}{m^\star z}$. Each $m^\star$ speficies a scheme (such as $\overline{MS}$).
3. γ_O^1: the LO anomalous dimension.
4. REG resumed form of the perturbative series calculated through Feynman diagrams.

General Features of Marginal PT

- Anomalous dimension $\gamma_0(\alpha)$ has no explicit ln μ dependence. Single log evolution.
- Requires specific singularity structure of Feynman integrals. Key structure: Natural factorization in each *Hepp sector* through maximal-forest of UV singularity.
- Borel non-summable. Renormalon (small number of diagrams, "bubble-chains") && instantons (large number of diagrams).
- UV-IR conspiracy between leading-power and high-power. Renormalon cancellation. Example: pole mass vs linear divergence in HQET.

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- Threshold asymptotics of quark-bilinear coefficient function
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More marginal asymptotics: Bjorken limit

- Next simplest object. Structure functions . Not completely Euclidean.

$$F(z^2, \lambda = -zv \cdot P) = \langle P | \bar{\psi}(zv) v \cdot \gamma [zv, 0] \psi(0) | P \rangle_c \quad (v^2 = -1.)$$

1. $F(z^2, \lambda) = \int_{-1}^{1} F(z^2, \alpha) e^{-i\lambda \cdot \alpha} d\alpha$, $F_N(z^2) = \int_{-1}^{1} \alpha^N F(z^2, \alpha) d\alpha$.

- The Bjorken limit : $-z^2 \to 0$ at fixed λ .

1. The limit can still be controlled by single-log asymptotics.
2. $F_N(z^2) \sim \mathcal{H}_N(\alpha(z)) \exp\left(\int_{\alpha(\mu)}^{\alpha(z)} \frac{\gamma_N(\alpha) - 2\gamma_F(\alpha)}{\beta(\alpha)} d\alpha\right) O_N(\mu) (1 + O(z^2 m^2))$
3. Coefficient function (partonic) && Leading-twist matrix elements.

The threshold-like asymptotics

- Now, consider the coefficient function $\mathcal{H}(z^2, i\lambda)$ && $\mathcal{H}_N(z^2)$.
- In the threshold limit $\lambda \to +\infty$ && $N \to +\infty$ one has ``double-log” type asymptotics.

1. The threshold limit is UV.
2. “Sudakov double-logarithms”

$$\frac{1}{\alpha(z_N)} \ln \alpha(z_N) \sim -\ln N \ln\ln N.$$

3. $-\gamma_N \to 2\Gamma_{cusp} \ln N e^{\gamma_E} + \gamma_{s+V}$

$$\mathcal{H}_N(z^2, \alpha(\mu)) = H^2_{\rm HL}\Big(0, \alpha(z_N)\Big) J(0, \alpha(z)) \exp\Big(\hat{O}_H(\alpha(z_N)) + \hat{O}_J(\alpha(z))\Big)$$
$$\times \exp\left(2\ln N e^{\gamma_E}\left(f_\Gamma(\alpha(\mu)) - \frac{\gamma_0}{\beta_0}\ln\alpha(\mu)\right) + f_{s+V+2F}(\alpha(\mu)) - \frac{\gamma^0_{s+V+2F}}{\beta_0}\ln\alpha(\mu)\right) . \tag{3.12}$$

In the above, the μ independent re-summation factors $\hat{O}_H$ and $\hat{O}_J$ are naturally written in terms of $\alpha(z_N) \equiv \alpha\left(\frac{2N}{|z|}\right)$ and $\alpha(z) \equiv \alpha\left(\frac{2e^{-\gamma_E}}{|z|}\right)$ as

$$\hat{O}_H(\alpha(z_N)) = -\frac{2\gamma_0}{\beta_0^2\alpha(z_N)}\ln\frac{1}{e\alpha(z_N)} + \frac{\beta_1\gamma_0}{\beta_0^3}\ln^2\alpha(z_N) + \frac{2f^0_\Gamma + \gamma^0_H}{\beta_0}\ln\alpha(z_N)$$
$$+ \frac{2\gamma_0}{\beta_0}\ln\alpha(z_N)k_\beta(\alpha(z_N)) + \frac{2\gamma_0}{\beta_0}l_\beta(\alpha(z_N)) - 2g_\Gamma(\alpha(z_N)) - f_H(\alpha(z_N)) ,$$
$$\hat{O}_J(\alpha(z)) = \frac{2\gamma_0}{\beta_0^2\alpha(z)}\ln\frac{1}{e\alpha(z)} - \frac{\beta_1\gamma_0}{\beta_0^3}\ln^2\alpha(z) - \frac{2f^0_\Gamma + \gamma^0_J}{\beta_0}\ln\alpha(z)$$
$$- \frac{2\gamma_0}{\beta_0}\ln\alpha(z)k_\beta(\alpha(z)) - \frac{2\gamma_0}{\beta_0}l_\beta(\alpha(z)) + 2g_\Gamma(\alpha(z)) + f_J(\alpha(z)) . \tag{3.13}$$

The HL Sudkov hard kernel

- The crucial object : the heavy-light Sudakov hard kernel.
- External : $p^2 = 0$ && $v^2 = -1$ (heavy-gauge-link to infinity)
- $H_{HL} \equiv H_{HL}(\ln\frac{2ip\cdot v}{\mu}, \alpha(\mu))$ depends on $L_p = \ln\frac{2ip\cdot v}{\mu}$.
- Double-log evolution: $\mu\frac{d\ln H_{HL}}{d\mu} = 2\Gamma_{cusp}L_p + 2\gamma_F + \gamma_V + 2\gamma_{HL} - \gamma_S$.
- γ_F: heavy-light current. γ_V: light-light Sudakov. γ_{HL}: heavy-light Wilson-line cusp. γ_S: light-light Wilson-line cusp.
- Know to NNLO.

The threshold soft factor

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- One needs also the threshold soft factor:
- $J\big(l_z, \alpha(\mu)\big) = \langle\Omega|T[zv + \infty n^+, zv][zv, 0][0, -\infty n^+]|\Omega\rangle.$
- $n^+ = \frac{1}{\sqrt{2}}(1,0,0,1)$ is the light-front plus direction.
- $\mu \frac{d\ln J}{d\mu} = 2\Gamma_{cusp}\, ln \frac{e^{\gamma_E}\mu|z|}{2} - 2\gamma_{HL} + 2\gamma_s.$
- Relates to another time-like heavy-quark jet function at NNLO through analytical continuation.

The threshold soft factor

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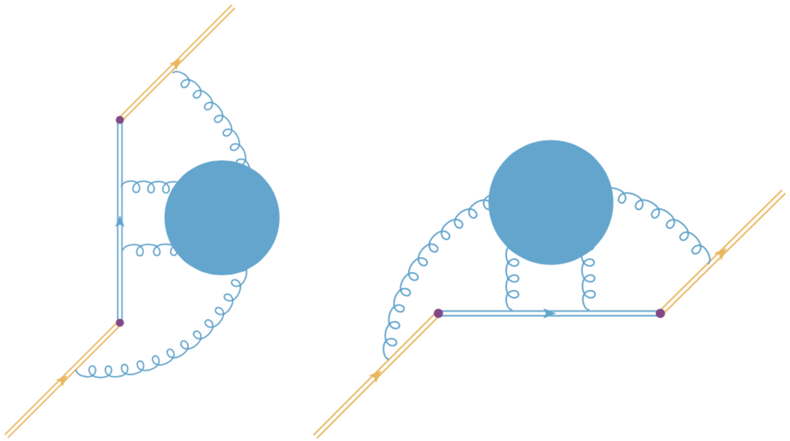
- The threshold soft factor: time-like (left) vs space-like (right).



- Re-sums ``soft exchanges" at scale $z^2 \gg \frac{\lambda^2}{z^2}$.

Threshold factorization

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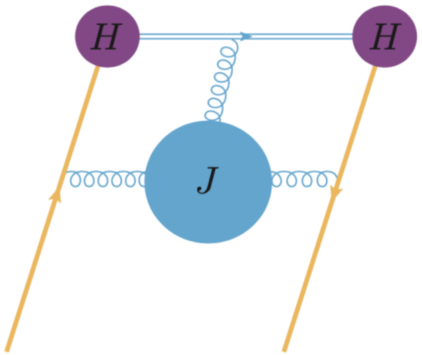
- We have checked that up to NNLO, one has the threshold factorization

$$\mathcal{H}_N(z^2,\alpha(\mu))|_{N\to\infty} \to H_{\rm HL}^2\left(\ln\frac{2N}{\mu|z|},\alpha(\mu)\right)J\left(l_z,\alpha(\mu)\right)\left(1+\mathcal{O}\left(\frac{1}{N}\right)\right)$$

$$\mathcal{H}\left(z^2,\lambda,\alpha(\mu)\right) = e^{-i\lambda}H_{\rm HL}^2\left(\ln\frac{-2i\lambda}{|z|\mu},\alpha(\mu)\right)J\left(l_z,\alpha(\mu)\right).$$



- The crucial feature: the threshold scale ($\frac{N}{z}$ or $\frac{\lambda}{z}$) is hard .

Threshold factorization

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- Given the RGEs, one has the fully factorized form of the threshold limit (moment space as an example):

$$\mathcal{H}_N(z^2,\alpha(\mu)) = H^2_{\rm HL}\Big(0,\alpha(z_N)\Big)J(0,\alpha(z))\exp\Big(\hat{O}_H(\alpha(z_N))+\hat{O}_J(\alpha(z))\Big)$$
$$\times\exp\left(2\ln Ne^{\gamma_E}\left(f_\Gamma(\alpha(\mu))-\frac{\gamma_0}{\beta_0}\ln\alpha(\mu)\right)+f_{s+V+2F}(\alpha(\mu))-\frac{\gamma^0_{s+V+2F}}{\beta_0}\ln\alpha(\mu)\right)\,. \quad (3.12)$$

In the above, the μ independent re-summation factors $\hat{O}_H$ and $\hat{O}_J$ are naturally written in terms of $\alpha(z_N)\equiv\alpha\left(\frac{2N}{|z|}\right)$ and $\alpha(z)\equiv\alpha\left(\frac{2e^{-\gamma_E}}{|z|}\right)$ as

$$\hat{O}_H(\alpha(z_N)) = -\frac{2\gamma_0}{\beta_0^2\alpha(z_N)}\ln\frac{1}{e\alpha(z_N)}+\frac{\beta_1\gamma_0}{\beta_0^3}\ln^2\alpha(z_N)+\frac{2f^0_\Gamma+\gamma^0_H}{\beta_0}\ln\alpha(z_N)$$
$$+\frac{2\gamma_0}{\beta_0}\ln\alpha(z_N)k_\beta(\alpha(z_N))+\frac{2\gamma_0}{\beta_0}l_\beta(\alpha(z_N))-2g_\Gamma(\alpha(z_N))-f_H(\alpha(z_N))\,,$$
$$\hat{O}_J(\alpha(z)) = \frac{2\gamma_0}{\beta_0^2\alpha(z)}\ln\frac{1}{e\alpha(z)}-\frac{\beta_1\gamma_0}{\beta_0^3}\ln^2\alpha(z)-\frac{2f^0_\Gamma+\gamma^0_J}{\beta_0}\ln\alpha(z)$$
$$-\frac{2\gamma_0}{\beta_0}\ln\alpha(z)k_\beta(\alpha(z))-\frac{2\gamma_0}{\beta_0}l_\beta(\alpha(z))+2g_\Gamma(\alpha(z))+f_J(\alpha(z))\,. \quad (3.13)$$

Renormalon in the HL hard kernel

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- A new feature of the threshold expansion for $\mathcal{H}_N$ is the presence of linear renormalon for the leading threshold-power hard kernel H_{HL}.
- Direct computation for bubble chain diagrams gives ($s = (n+1)\epsilon$)

$$V_n = \frac{\alpha^{n+1} C_F}{2} \left(\frac{\beta_0}{2}\right)^n \frac{V(\epsilon, (n+1)\epsilon)}{(n+1)^2 \epsilon^{n+2}} ,$$

$$V(\epsilon, s) = e^{s\gamma_E} \left(\frac{i\sigma\mu}{2p^z}\right)^{2s} f(\epsilon)^{\frac{s}{\epsilon}-1} \frac{(\epsilon-1)s\Gamma[1-s]}{\sin[2\pi s]\Gamma[2-s-\epsilon]} .$$

- The $\sin 2\pi s$ leads to a linear renormalon at $s = \frac{1}{2}$.

Renormalon in the HL hard kernel

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- After renormalization one obtains the Borel-transform:

$$B(u) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{RV_n}{\alpha^{n+1}} = R(u) + \frac{C_F}{2} \sum_{n=0}^{\infty} \frac{u^n}{(n+2)!} \frac{d^{n+2}V_0(u)}{du^{n+2}}|_{u=0}$$
$$= R(u) + \frac{C_F}{2} \frac{V_0(u) - uV_0'(0) - V_0(0)}{u^2} ,$$

$$V_0(u) = \frac{1}{2} \frac{ue^{\frac{5u}{3}}}{u-1} \left(\frac{1}{\sin \pi u} + \frac{i\text{sign}(z)}{\cos \pi u} \right) \left(\frac{\mu^2}{4p_z^2} \right)^u .$$

- A pole at $u = \frac{1}{2}$. What is its role in the threshold expansion ?

Renormalon in the threshold expansion

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- One can investigate the bubble chain diagram for the coefficient functions.
- Result known before (Braun.2018). Only ``Sudakov-diagram” requires attention ($s = (n+1)\epsilon$). No linear renormalon!

$$\mathfrak{h}_n(z^2, i\lambda) = e^{\lambda} \frac{2\Gamma(1-s)\Gamma(2s)}{(4\pi)^{\frac{D}{2}}} \left(\frac{z^2}{4}\right)^s \Big({}_2\tilde{F}_2(1, 2s; -\epsilon + s + 2, 2s+1; -\lambda)$$

$$+ 2\lambda \left({}_2\tilde{F}_2(1, 2s+1; -\epsilon + s + 2, 2s+2; -\lambda) - {}_2\tilde{F}_2(2, 2s+1; -\epsilon + s + 3, 2s+2; -\lambda) \right) \Big) .$$

$$\mathfrak{h}_{n,N}(z^2) = -\frac{2\Gamma(-s)}{(4\pi)^{\frac{D}{2}} \Gamma(s-\epsilon+1)} \left(\frac{z^2}{4}\right)^s \Big(N I(2s, s-\epsilon+1; N-1) + s I(2s-1, s-\epsilon; N) \Big) .$$

$$I(a,b;N) \equiv \int_0^1 t^a dt \int_0^1 x^b (1-\bar{x}t)^N dx$$

$$= -\Gamma(N+1)\Gamma(-a) \left(\frac{{}_3F_2(1,-a,-b;1-a,N+2;1)}{\Gamma(1-a)\Gamma(N+2)} - \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(-a+b+1)\Gamma(a+N+2)} \right) .$$

Renormalon in the threshold expansion 2311.006907, Liu & Su

- One can now perform the threshold expansion in both spaces.

$$\mathfrak{h}_n(z^2, i\lambda) = \frac{2\pi e^{\lambda}}{(4\pi)^{\frac{D}{2}}}\left(\frac{|z|}{2\lambda}\right)^{2s}\frac{(1-\epsilon)\Gamma(-s)}{\sin 2\pi s\Gamma(2-s-\epsilon)}$$
$$-\frac{e^{\lambda}}{(4\pi)^{\frac{D}{2}}}\left(\frac{z^2}{4}\right)^{s}\left(\frac{\Gamma(-s)}{s\Gamma(1-\epsilon+s)}+\frac{2(\epsilon-1)\Gamma(-s)}{(2s-1)\Gamma(1-\epsilon+s)\lambda}+\mathcal{O}\left(\frac{1}{\lambda^2}\right)\right).$$

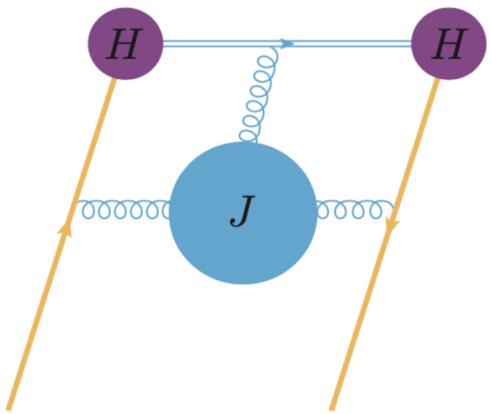
$$\mathfrak{h}_{n,N}(z^2) = \frac{2\pi}{(4\pi)^{\frac{D}{2}}}\left(\frac{z^2}{4N^2}\right)^{s}\frac{(1-\epsilon)\Gamma(-s)}{\sin 2\pi s\Gamma(2-s-\epsilon)}\left(1-\frac{s(2s+1)}{N}+\mathcal{O}\left(\frac{1}{N^2}\right)\right)$$
$$-\frac{1}{(4\pi)^{\frac{D}{2}}}\left(\frac{z^2}{4}\right)^{s}\left(\frac{\Gamma(-s)}{s\Gamma(1-\epsilon+s)}+\frac{2(\epsilon-1)\Gamma(-s)}{(2s-1)\Gamma(1-\epsilon+s)N}+\mathcal{O}\left(\frac{1}{N^2}\right)\right).$$

- The first term is exactly the HL Sudakov hard kernel. (Linear renormalon).
- The second term is the LP threshold soft factor. (No linear renormalon.)

Renormalon cancellation between LP and NLP

- The term at next-to-leading threshold power is:

$$\mathfrak{h}_n^{\rm NLP}(z^2, i\lambda) = -\frac{e^\lambda}{\lambda}\frac{2}{(4\pi)^{\frac{D}{2}}}\left(\frac{z^2}{4}\right)^s \frac{(\epsilon-1)\Gamma(-s)}{(2s-1)\Gamma(1-\epsilon+s)} \,.$$



- It cancels the $s = \frac{1}{2}$ renormalon for the LP HL hard kernel

$$\frac{2\pi e^\lambda}{(4\pi)^{\frac{D}{2}}}\left(\frac{|z|}{2\lambda}\right)^{2s} \frac{(1-\epsilon)\Gamma(-s)}{\sin 2\pi s\Gamma(2-s-\epsilon)}$$

- This is due to the fact that threshold scale is $\frac{z}{|\lambda|} = \frac{1}{|p\cdot v|}$. Otherwise, it will not cancel.

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The NLP threshold soft factor

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- What is the nature of the NLP contribution?
- At level of bubble-chain diagram. Only one operator:

$$J_q(z,\alpha(\mu)) = \frac{i}{2}\langle\Omega|\infty n^+ + zn_z, zn_z][zn_z, 0]\int_{-\infty}^{0} dx^+[0,x^+]D_\perp^2[x^+,-\infty]|\Omega\rangle \ .$$

- Sub-eikonal at external quark

$$\frac{i(p^+ + k^+)}{(p+k)^2 + i0} = \frac{i}{2(k^- + i0)} + \frac{i}{p^+}\frac{k_\perp^2}{4(k^-)^2} + \mathcal{O}\left(\frac{1}{p^+}\right)^2 \ .$$

The NLP threshold soft factor: UV renormalon

- One can show the bubble-chain diagram for J_q exactly reproduces the NLP contribution.

$$\mathfrak{h}_n^{\rm NLP}(z^2, i\lambda) = -\frac{e^{\lambda}}{\lambda}\frac{2}{(4\pi)^{\frac{D}{2}}}\left(\frac{z^2}{4}\right)^s \frac{(\epsilon-1)\Gamma(-s)}{(2s-1)\Gamma(1-\epsilon+s)} .$$

- The operator for J_q is dimensional 1→ possible linear UV divergence.
- Introducing a cutoff UV regulator $a > 0$, one has

$$e^{i\lambda}J_n(z,a) = \frac{i}{(4\pi)^{\frac{D}{2}}p^z}\left(\frac{\sqrt{\pi}(\epsilon-1)2^{\frac{3}{2}-3s}a^{2s-1}\Gamma\left(\frac{1}{2}-s\right)}{\sin\pi s\Gamma(-\epsilon+s+1)}\right) - \frac{i}{zp^z}\frac{2}{(4\pi)^{\frac{D}{2}}}\left(\frac{z^2}{4}\right)^s\frac{(\epsilon-1)\Gamma(-s)}{(2s-1)\Gamma(-\epsilon+s+1)} + \mathcal{O}(az) .$$

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The NLP threshold soft factor: UV renormalon

- Introducing a cutoff UV regulator $\boldsymbol{a > 0}$, one has

$$e^{i\lambda}J_n(z,a) = \frac{i}{(4\pi)^{\frac{D}{2}}p^z}\left(\frac{\sqrt{\pi}(\epsilon-1)2^{\frac{3}{2}-3s}a^{2s-1}\Gamma\left(\frac{1}{2}-s\right)}{\sin\pi s\Gamma(-\epsilon+s+1)}\right) - \frac{i}{zp^z}\frac{2}{(4\pi)^{\frac{D}{2}}}\left(\frac{z^2}{4}\right)^s\frac{(\epsilon-1)\Gamma(-s)}{(2s-1)\Gamma(-\epsilon+s+1)} + \mathcal{O}(az)\ .$$

- The firs term is ``linear UV divergence". The second term is ``finite part".
- The $s = \frac{1}{2}$ singularity cancel between them.
- Thus the linear renormalon of the NLP soft factor is *UV renormalon.*

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Physical interpretations

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- The scale separation in the threshold limit: $\left|\frac{\lambda}{z}\right| \gg \frac{1}{|z|} \gg \Lambda_{QCD}$.
- Linear renormalon is due to the ambiguity of separating the first two hard scales. IR with respect to $\left|\frac{\lambda}{z}\right|$ but UV with respect to $\frac{1}{|z|}$.
- Genuine non-perturbative indicator: threshold soft factors. Only quadratic or higher. Consistent with Braun.2018.
- High renormalon only leads to $\frac{z^n \Lambda_{QCD}^n}{\lambda^m}$ with $m \geq 1$. No enhancement of non-perturbative effect in the threshold limit.

A conjecture.

- High renormalon only leads to $\frac{z^n \Lambda_{QCD}^n}{\lambda^m}$ with $m \geq 1$. No enhancement of non-perturbative effect in the threshold limit.
- This leads to the conjecture: the threshold expansion and twist expansion for coordinate-space correlators commute.
- I can show this in the 2D large N Gross-Neveu.

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2D Gross Neveu

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- Exact calculation at $\frac{1}{N}$ using a trick in Braun/Beneke 1998 (O(n) model).

$$
\begin{aligned}
&\langle p,i|\bar{\psi}^i(x)\psi^i(0)|p,i\rangle - \langle p,i|\bar{\psi}^j(x)\psi^j(0)|p,i\rangle \\
&\equiv F(x,p) = \bar{u}(p)u(p)f_a(z^2m^2,\lambda) + m\bar{u}(p)\not{z}u(p)f_b(z^2m^2,\lambda)\ , \qquad (2.1)
\end{aligned}
$$

- Twist expansion can be performed exactly ($f_a = -F_1 + F_2$)

$$
\begin{aligned}
F_1(z^2m^2,\lambda) =& \sum_{l=0}^{\infty}\left(\frac{z^2m^2}{4}\right)^l q_1^{(l)}(\lambda,\mu) \\
&+ \sum_{l=0}^{\infty}\left(\frac{z^2m^2}{4}\right)^l \int_0^{\infty} dt \sum_{p=0}^{\infty}\left(\frac{z^2m^2}{4}\right)^p \left(\mathcal{H}_1^{l,p}(t,\alpha(z),\lambda,\mu) + q_1^{l,p}(t,\lambda,\mu)\right)\ ,
\end{aligned}
$$

2D Gross Neveu: Twist expansion.

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- Renormalon all cancels. Converges absolutely for any $z^2 > 0$.

$$F_1(z^2m^2,\lambda) = \sum_{l=0}^{\infty}\left(\frac{z^2m^2}{4}\right)^l q_1^{(l)}(\lambda,\mu) + \sum_{l=0}^{\infty}\left(\frac{z^2m^2}{4}\right)^l \int_0^{\infty} dt \sum_{p=0}^{\infty}\left(\frac{z^2m^2}{4}\right)^p \left(\mathcal{H}_1^{l,p}(t,\alpha(z),\lambda,\mu) + q_1^{l,p}(t,\lambda,\mu)\right),$$

$$\mathcal{H}_1^{l,p}(t,\alpha(z),\lambda) = \frac{(-1)^{l+p}\Gamma(2l+2)\Gamma(2t+2p+1)\Gamma(-l-p-t)}{4\pi l!p!\Gamma(2t+p+1)} \times {}_1\tilde{F}_1(2+2l,1+2l+p+t,-\lambda)\left(\frac{z^2m^2}{4}\right)^t,$$

$$e^{-\lambda}q_1(-\lambda) = \int_0^1 d\alpha\left(\frac{(1-\alpha)^2}{4\pi\alpha^2\ln^2\alpha}\right)_+ \frac{1-\alpha^2}{(1+\alpha)^3}e^{-\lambda\alpha} + \int_0^1 d\alpha\left(\frac{(1-\alpha)^2}{4\pi\alpha^2\ln\alpha}\right)_+ \frac{1+4\alpha+\alpha^2}{(1+\alpha)^3}e^{-\lambda\alpha} + \frac{\lambda+1}{4\pi}\left(\gamma_E - 3\ln 2\right) - \frac{\ln 2}{2\pi}.$$

$$q_1^{l,p}(t,\lambda) = \frac{(-1)^p\Gamma(2l+2)\Gamma(2p+1)}{4\pi l!(l+p)!}\frac{\Gamma(t-p)}{\Gamma(t+p+1)}\,{}_1\tilde{F}_1(2+2l,1+2l+p,-\lambda).$$

2D Gross Neveu: threshold expansion

Liu, 2403.06787

- Threshold expansion can also be performed:

$$F_1^s(z^2m^2, s, \lambda) \sim \frac{1}{4\pi}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\Gamma(-l-s)\frac{(-1)^{k+l}}{k!l!}\frac{\Gamma(k+2l+2s+2)}{\Gamma(-k-s-1)}\lambda^{-k-2l-2-2s}\left(\frac{z^2m^2}{4}\right)^l .$$

$$F_1^h(z^2m^2, s, \lambda) \sim \frac{1}{4\pi}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\Gamma(s-l)\frac{(-1)^{k+l}}{k!l!}\frac{\Gamma(k+2l+2)}{\Gamma(-k-s-1)}\lambda^{-k-2l-2}\left(\frac{z^2m^2}{4}\right)^{-s+l} .$$

- Only thing to worry: $(\lambda^2)^{-s}$. But no problem!

2D Gross Neveu: threshold expansion

Liu, 2403.06787

- Terms with (λ^{-2s}) has no enhancement at large λ:

$$q_1^{k=0}(\lambda) = -\frac{1}{4\pi\lambda^2}\int_0^\infty dt \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} \frac{(s+1)\Gamma(1-2s)\Gamma(2s+2)\Gamma(s+t)}{\Gamma(-s+t+1)} (\lambda^2)^{-s} .$$

$$q_1^{k=0}(\lambda) = -\frac{1}{4\pi\lambda^2}\left(\frac{1}{2} - \frac{1}{2\lambda^2} + \frac{11}{4\lambda^4} - \frac{191}{6\lambda^6} + \mathcal{O}\left(\frac{1}{\lambda^8}\right)\right) .$$

- Thus, power $(z^2m^2)^l$ in F_1^s decay at large λ at most λ^{-2-2l}.
- No conflict between threshold and twist expansion!

Threshold expansion is divergent!

2311.006907, Liu & Su

- Unlike the twist expansion, the threshold expansion is only asymptotic.
- Even at the bubble chain level, one can see this (back to QCD)

$$e^{-\lambda}\mathfrak{h}_n(z^2,i\lambda)=\frac{2}{(4\pi)^{\frac{D}{2}}}\left(\frac{|z|}{2}\right)^{2s}\left(\frac{\pi}{\lambda^{2s}}\frac{(1-\epsilon)\Gamma(-s)}{\sin 2\pi s\Gamma(2-s-\epsilon)}\right.$$
$$\left.-\frac{\Gamma(-s)}{2s\Gamma(-\epsilon+s+1)}+\sum_{k=1}^{\infty}\frac{(-1)^k(\epsilon-1)\Gamma(-s)}{(2s-k)\Gamma(s-\epsilon+2-k)\lambda^k}\right)$$
$$+\frac{2e^{-\lambda}}{(4\pi)^{\frac{D}{2}}}\left(\frac{|z|}{2}\right)^{2s}(-\lambda)^{\epsilon-s}\left(\sum_{k=1}^{\infty}\frac{(1-\epsilon)\Gamma(\epsilon+s-1)\Gamma(-s)}{\Gamma(\epsilon+s-k)\lambda^{k+1}}\right),$$

- Exponentially small terms due to ``small-x" limit.
- Resurgent relation between threshold and ``small-x" asymptotics !

Summary

- Marginal Sudakov asymptotics through quark-bilinear coefficient function in threshold limit.
- Threshold scale is UV. $\left|\frac{\lambda}{z}\right| \gg \frac{1}{|z|} \gg \Lambda_{QCD}$. Ambiguity of separating the two hard scales: linear threshold renormalon.
- Cancel with UV renormalon of NLP threshold soft factor.
- No enhancement of non-perturbative effect in the threshold limit.
- Threshold expansion commute with twist expansion: exact verification in large N Gross-Neveu.
- Divergence of threshold expansion. Resurgent relation between small-x limit and threshold limit.