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MAX-PLANCK-INSTITUT
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electron-muon scattering at NNLO

William J. Torres Bobadilla
Max-Planck-Institut Für Physik

MITP TOPICAL WORKSHOP
The Evaluation of the Leading Hadronic Contribution to the Muon g-2: Toward the MUonE Experiment
14 – 18 November 2022
https://indico.mitp.uni-mainz.de/event/248
Mainz Institute for Theoretical Physics

The muon g-2: the QED contribution

$a_\mu^{QED} = (1/2)(\alpha/\pi)$ Schwinger 1948

$+ 0.765857426\,(16)\,(\alpha/\pi)^2$

Sommerfield; Petermann; Suura&Wichmann '57; Elend '66; MP '04

$+ 24.05050988\,(28)\,(\alpha/\pi)^3$

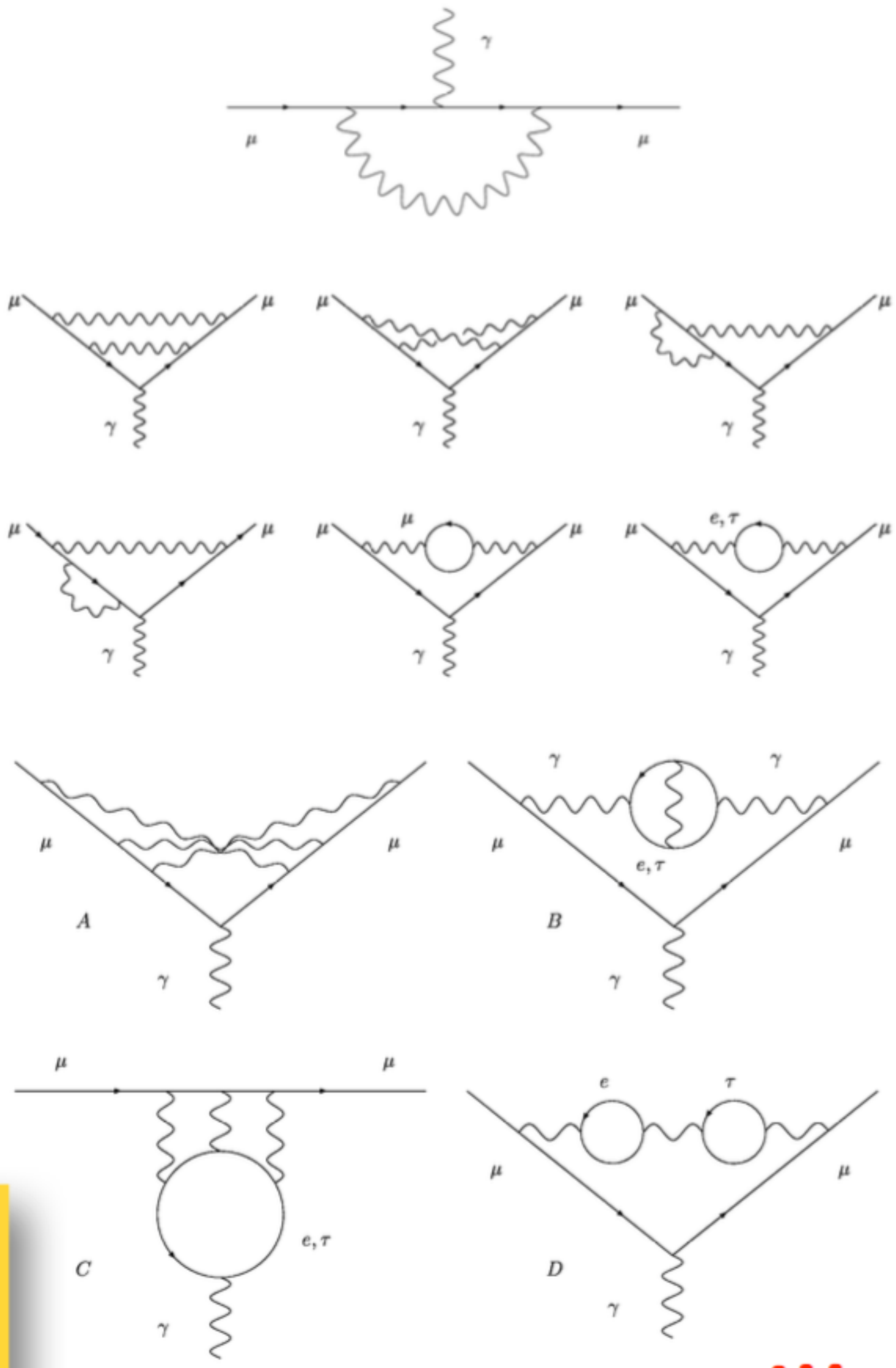
Remiddi, Laporta, Barbieri … ; Czarnecki, Skrzypek; MP '04;
Friot, Greynat & de Rafael '05, Mohr, Taylor & Newell 2012

$+ 130.8780\,(60)\,(\alpha/\pi)^4$

Kinoshita & Lindquist '81, … , Kinoshita & Nio '04, '05;
Aoyama, Hayakawa,Kinoshita & Nio, 2007, Kinoshita et al. 2012 & 2015;
Steinhauser et al. 2013, 2015 & 2016 (all electron & τ loops, analytic);
S. Laporta, PLB 2017 (mass independent term). COMPLETED²!

$+ 750.80\,(89)\,(\alpha/\pi)^5$ **COMPLETED!**

Kinoshita et al. '90, Yelkhovsky, Milstein, Starshenko, Laporta,…
Aoyama, Hayakawa, Kinoshita, Nio 2012 & 2015 & 2017



…

Adding up, I get:

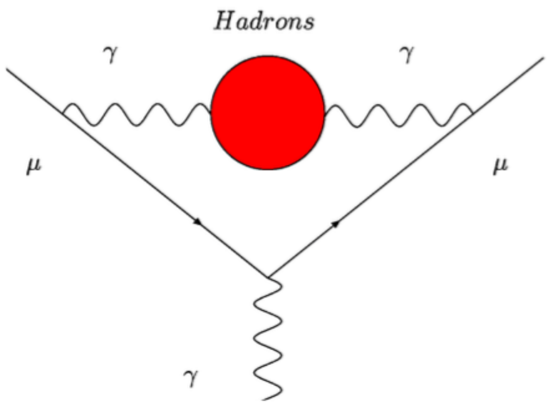
$a_\mu^{QED} = 116584718.932\,(20)(23) \times 10^{-11}$

from coeffs, mainly from 4-loop unc — from α (Cs)

with $\alpha = 1/137.035999046(27)$ [0.2ppb] 2018

New space-like proposal for HLO

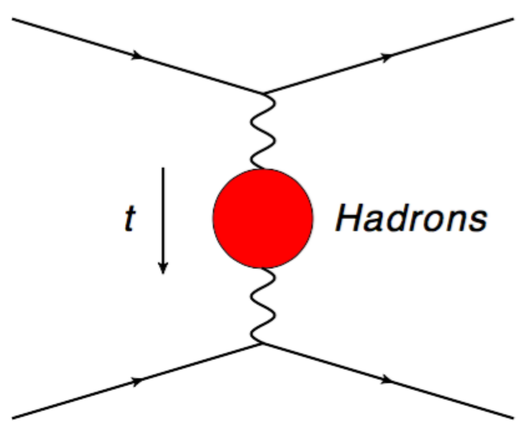
- At present, the leading hadronic contribution a_μ^{HLO} is computed via the time-like formula:



$$a_\mu^{\mathrm{HLO}} = \frac{1}{4\pi^3} \int_{4m_\pi^2}^{\infty} ds\, K(s)\, \sigma_{\mathrm{had}}^0(s)$$

$$K(s) = \int_0^1 dx\, \frac{x^2\,(1-x)}{x^2 + (1-x)\,(s/m_\mu^2)}$$

- Alternatively, exchanging the x and s integrations in a_μ^{HLO}



$$a_\mu^{\mathrm{HLO}} = \frac{\alpha}{\pi} \int_0^1 dx\,(1-x)\,\Delta\alpha_{\mathrm{had}}[t(x)]$$

$$t(x) = \frac{x^2 m_\mu^2}{x-1} < 0$$

$\Delta\alpha_{had}(t)$ is the hadronic contribution to the running of α in the space-like region. It can be extracted from scattering data!

M. Passera UZH Feb 4 2019

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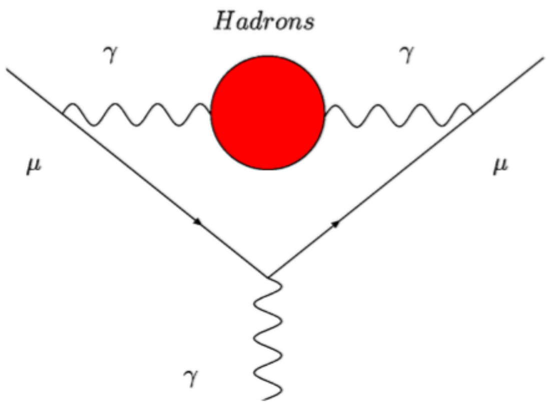

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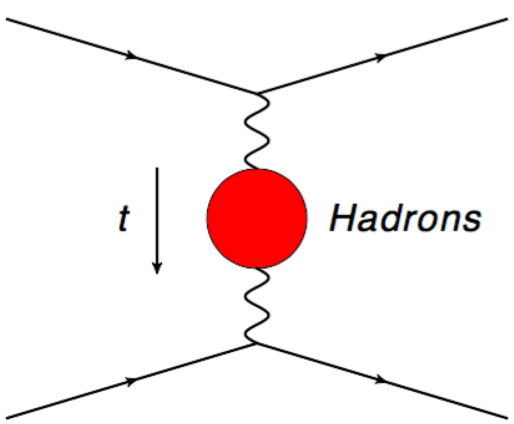

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space-like region. It can be extracted from scatterin[...]

M. Passera UZH Feb 4 2019

Muon-electron scattering:
The MUonE Project

Abbiendi, Carloni Calame, Marconi, Matteuzzi, Montagna,
Nicrosini, MP, Piccinini, Tenchini, Trentadue, Venanzoni
EPJC 2017 - arXiv:1609.08987

Outline

- Motivation
- $e^+e^- \to \mu^+\mu^-$ and $q\bar{q} \to t\bar{t}$ @ two loops
- IR pole predictions
- Results
- Outlook

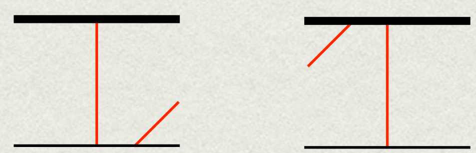
$e\mu$-scattering @ NLO

$$\mu^+(p_1) + e^-(p_2) \to \mu^+(p_3) + e^-(p_4)$$

Anatomy

- Born matrix element
  tree-level & n-pt process

- Real contribution
  Tree-level (n+1)-particles



- Virtual Contribution
  one-loop n-particles

- No assumption made in the newest result
- QED & EW effects
- Full lepton mass dependence
- Fully differential fixed order MC @ NLO

$$\hat{\sigma}_{NLO} \sim \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\hat{\sigma}^{R}_{NLO} + \int_{\mathrm{d}\Phi_m} \mathrm{d}\hat{\sigma}^{V}_{NLO}$$ + MC integration

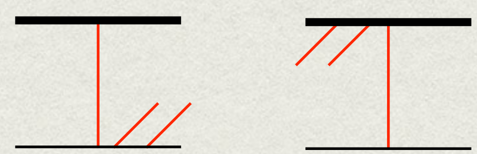
[Carloni Calame, Alacevich, Chiesa, Montagna, Nicrosini, Piccinini (2018)]

$e\mu$-scattering @ NNLO

$$\mu^+(p_1) + e^-(p_2) \to \mu^+(p_3) + e^-(p_4)$$

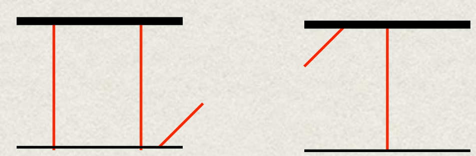
Anatomy

- Real-Real contribution
  Tree-level (n+2)-particles



[OpenLoops framework]

- Real-Virtual Contribution
  one-loop (n+1)-particles



- Virtual-Virtual Contribution
  two-loop n-particles

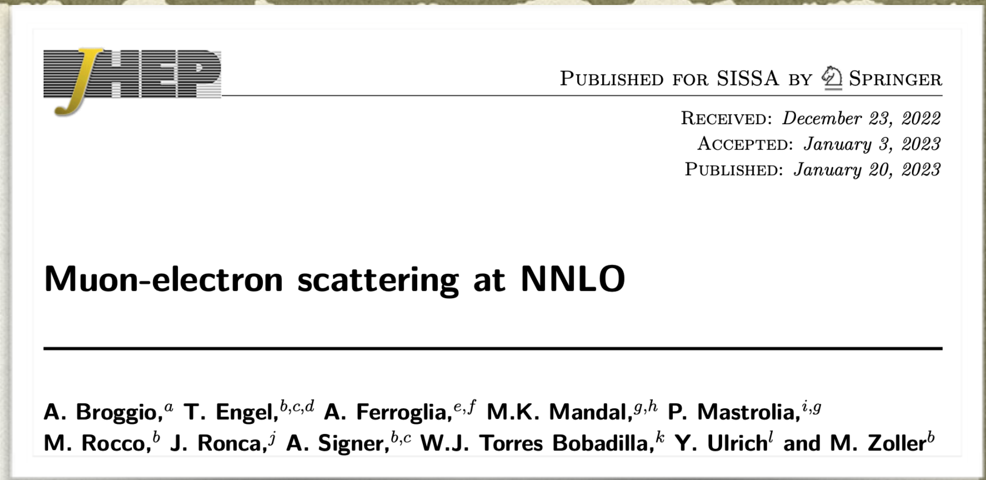
[Bonciani et al (2021)]

$$\hat{\sigma}_{NNLO} \sim \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\hat{\sigma}^{RR}_{NNLO} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\hat{\sigma}^{RV}_{NNLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\hat{\sigma}^{VV}_{NNLO}$$

+ Subtractions & MC integrations

[McMule framework]

$e\mu$-scattering @ NNLO



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Muon-electron scattering at NNLO

A. Broggio,[a] T. Engel,[b,c,d] A. Ferroglia,[e,f] M.K. Mandal,[g,h] P. Mastrolia,[i,g] M. Rocco,[b] J. Ronca,[j] A. Signer,[b,c] W.J. Torres Bobadilla,[k] Y. Ulrich[l] and M. Zoller[b]

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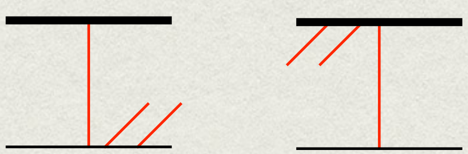
Anatomy

- Real-Real contribution
  Tree-level (n+2)-particles



[OpenLoops framework]

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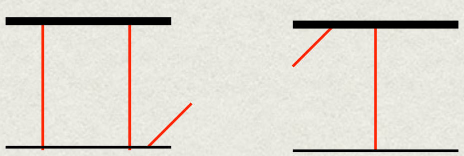

- Virtual-Virtual Contribution
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+ Subtractions & MC integrations

[McMule framework]

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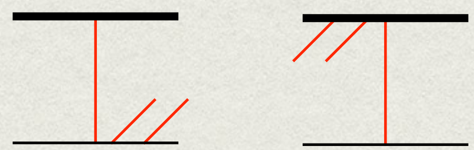
Anatomy

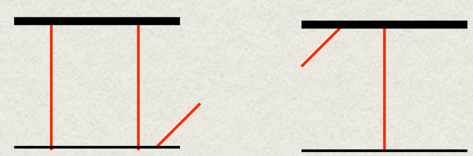
☆ **This talk**

- Real-Real contribution
  Tree-level (n+2)-particles



[OpenLoops framework]

- Real-Virtual Contribution
  one-loop (n+1)-particles



- Virtual-Virtual Contribution
  two-loop n-particles

[Bonciani et al (2021)]

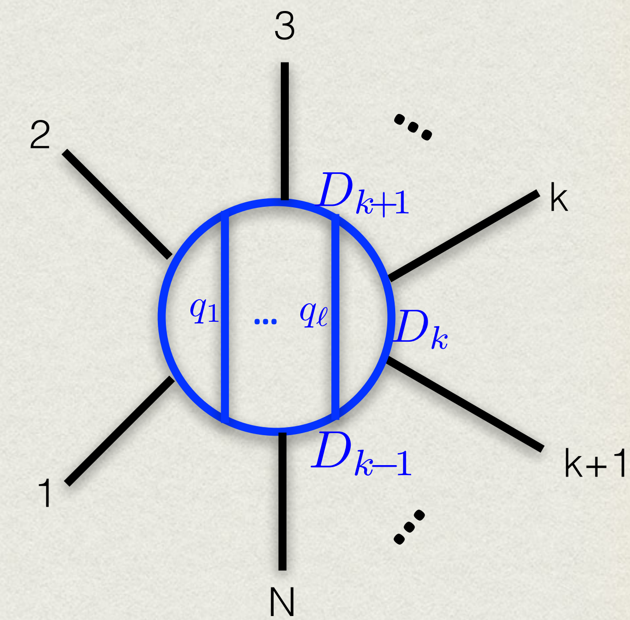
$$\hat{\sigma}_{NNLO} \sim \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\hat{\sigma}^{RR}_{NNLO} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\hat{\sigma}^{RV}_{NNLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\hat{\sigma}^{VV}_{NNLO}$$

+ Subtractions & MC integrations

[McMule framework]

Analytic evaluations

In loop calculations, one finds

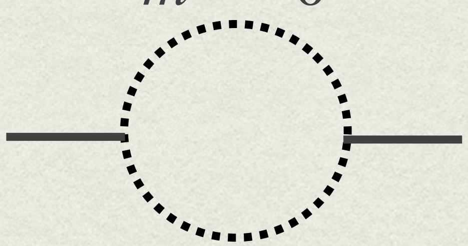


$$J_N^{(L),D}(1,\ldots,n;n+1,\ldots,m) = \int \prod_{i=1}^{L} \frac{d^D \ell_i}{\imath\, \pi^{D/2}} \frac{\prod_{k=n+1}^{m} D_k^{-\nu_k}}{\prod_{j=1}^{n} D_j^{\nu_j}}$$

$$D_i = q_i^2 - m_i^2 + \imath 0$$

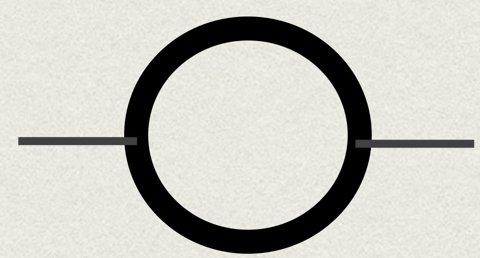
Complexity easily increases:

$m^2 = 0$



$$\frac{1}{\epsilon}(-p^2)^{-1-\epsilon}\left(-2 + \frac{\pi^2}{6}\epsilon^2 + \frac{14}{3}\zeta_3\epsilon^3 + \mathcal{O}\left(\epsilon^4\right)\right)$$

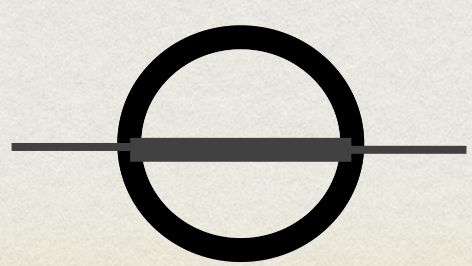
$m^2 \neq 0$



$$\frac{2}{\sqrt{(-p^2)(4m^2-p^2)}} \log\left(\frac{\sqrt{1-4m^2/p^2}+1}{\sqrt{1-4m^2/p^2}-1}\right) + \mathcal{O}(\epsilon)$$

—> squared roots

$m^2 \neq 0$



$$-\frac{4\mathrm{K}(\lambda)}{(p^2+m^2)\sqrt{a_{13}a_{24}}}\left[2\mathcal{E}_4\begin{pmatrix} 0 & -1 \\ 0 & \infty \end{pmatrix};1,\vec{a}\right) + \mathcal{E}_4\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix};1,\vec{a}\right) + \mathcal{E}_4\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix};1,\vec{a}\right)\right]$$

$$\mathrm{K}(\lambda) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\lambda x^2)}} \qquad \mathcal{E}_4\left(\begin{matrix} n_1 & \ldots & n_k \\ c_1 & \ldots & c_k \end{matrix};t,\vec{a}\right) = \int_0^x dt\,\Psi_{n_1}(c_1,t,\vec{a})\,\mathcal{E}_4\left(\begin{matrix} n_2 & \ldots & n_k \\ c_2 & \ldots & c_k \end{matrix};t,\vec{a}\right)$$

—> elliptic integrals

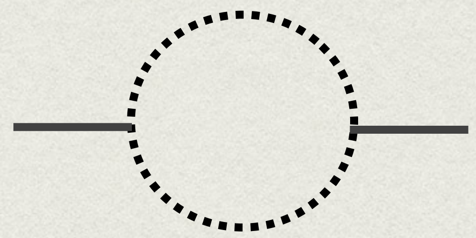
Algorithms for computing Feynman integrals

Standard approach

- DEQ :: Feynman integrals are not independent

$$\partial_x \overrightarrow{J}(x) = A_i(x,\epsilon)\,\overrightarrow{J}(x)$$

$m^2 = 0$



$$\frac{1}{\epsilon}(-p^2)^{-1-\epsilon}\left(-2+\frac{\pi^2}{6}\epsilon^2+\frac{14}{3}\zeta_3\epsilon^3+\mathcal{O}\left(\epsilon^4\right)\right)$$

Canonical form
Conjecture: there exist a basis of <u>uniform transcendental weight functions</u>

[Henn (2013)]

$$\partial_x \overrightarrow{g}(x) = \epsilon B(x)\,\overrightarrow{g}(x) \quad \longrightarrow \quad d\,\overrightarrow{g}(x,\epsilon) = \epsilon\left(d\tilde{B}\right)\overrightarrow{g}(x;\epsilon)$$

$$\tilde{B} = \sum_k B_k \log \alpha_k(x)$$

Uniform weight function

- Solution in terms of iterated integrals :: HPL/GPL (PolyLogs)

$$\mathcal{G}\left(a_1,\ldots,a_n;x\right) = \int_0^x dt\,\frac{1}{t-a_n}\,\mathcal{G}\left(a_1,\ldots,a_{n-1};t\right)$$

Numerical implementations:
`GinaC`, `HandyG`, `FastGPL`, …

$e^+e^- \to \mu^+\mu^-$ @ two loops

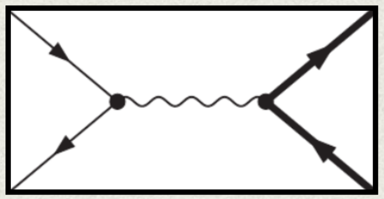
- emu scattering —> di-muon production
- Close connection to $q\bar{q} \to t\bar{t}$
  (completely known numerically in literature)

[Bonciani, Ferroglia, Gehrmann, Studerus (2009)]
[Barnreuther, Czakon, Fiedler (2013)]

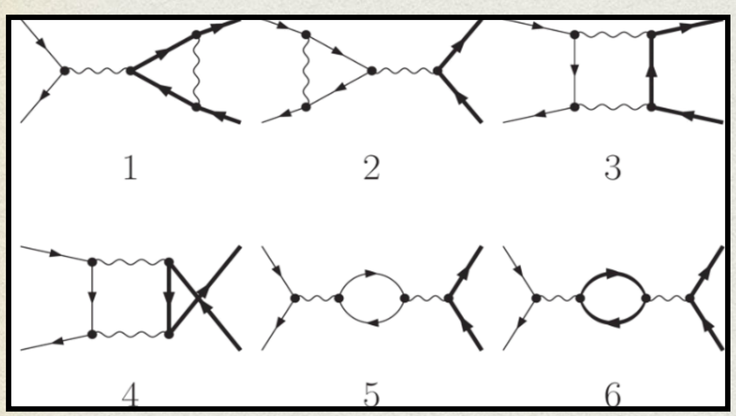
- Checks from QCD to QED

Anatomy of $e^+e^- \to \mu^+\mu^-$ (up-to two loops)

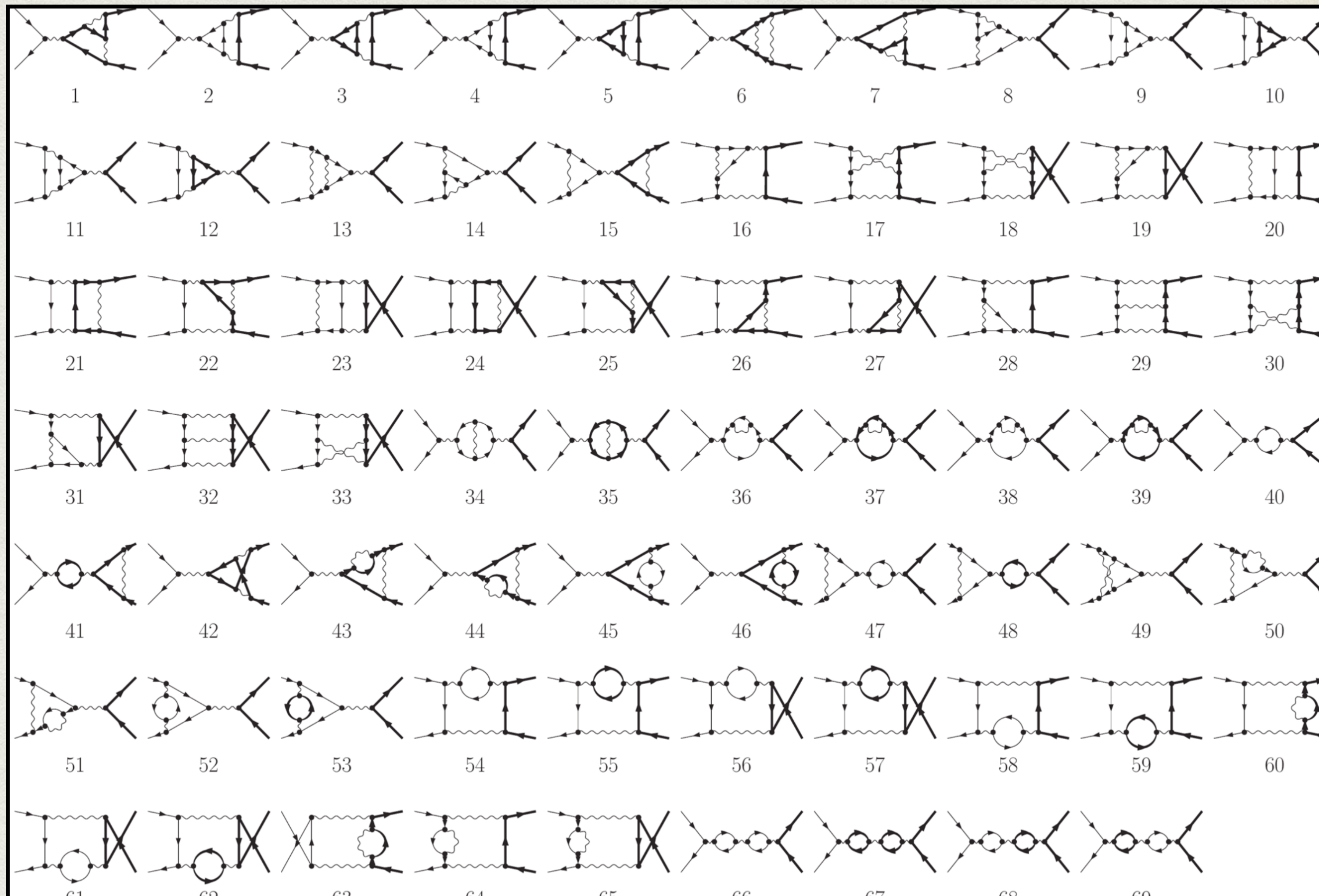
- Tree-level



- One-loop



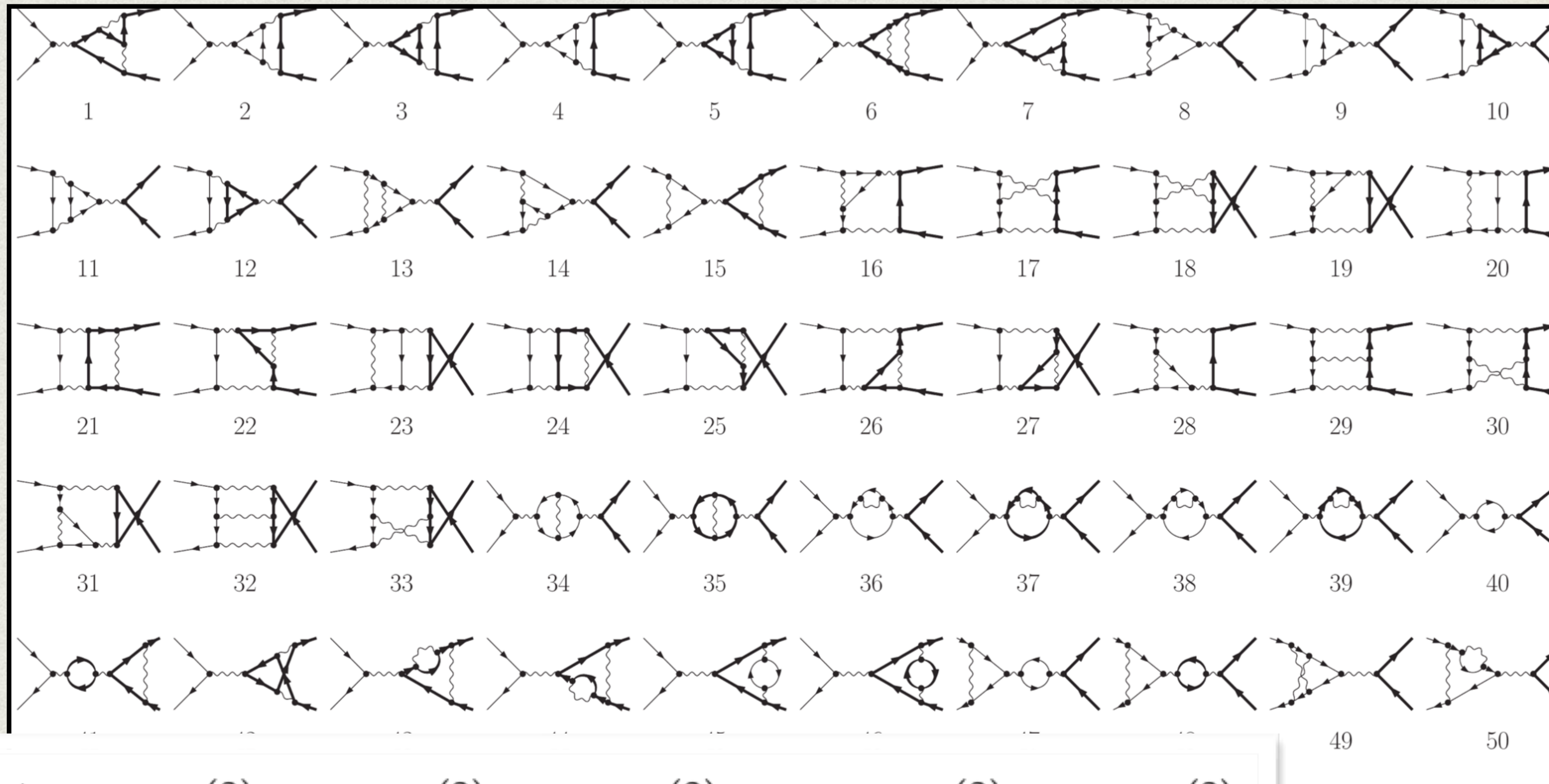
- Two-loop



Anatomy of $e^+e^- \to \mu^+\mu^-$ (up-to two loops)

- Two-loop

69 diagrams @ 2-loop



Automatically organised in groups

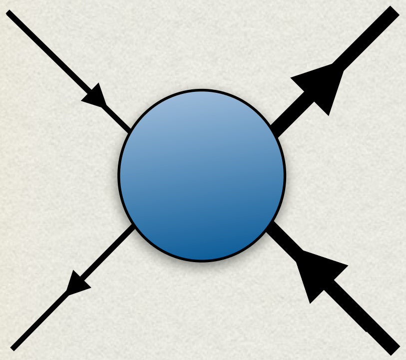
$$\mathcal{M}^{(2)} = A^{(2)} + \boldsymbol{n_l}\, B_l^{(2)} + \boldsymbol{n_h}\, C_h^{(2)} + \boldsymbol{n_l^2}\, D_l^{(2)} + \boldsymbol{n_h n_l}\, E_{hl}^{(2)} + \boldsymbol{n_h^2}\, F_h^{(2)}$$

Integrand reductions by means of **AIDA**

$$\mathcal{M}^{(2)} = \frac{\mathcal{M}^{(2)}_{-4}}{\epsilon^4} + \ldots + \frac{\mathcal{M}^{(2)}_{-1}}{\epsilon} + \mathcal{M}^{(2)}_0 + \mathcal{O}(\epsilon)$$

Algebraic decomposition

$$e^+e^- \to \mu^+\mu^-$$



$$\mathcal{A}(\alpha) = 4\pi\alpha \left[\mathcal{A}^{(0)} + \left(\frac{\alpha}{\pi}\right)\mathcal{A}^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 \mathcal{A}^{(2)} + \mathcal{O}\left(\alpha^3\right)\right]$$

$$s = (p_1+p_2)^2 \,, \quad t = (p_2 - p_3)^2 \,,$$
$$u = (p_1 - p_3)^2 \,, \quad s+t+u = 2M^2 \,.$$

- Compute interference

$$\mathcal{M}^{(n)}\left(e^+e^- \to \mu^+\mu^-\right) = \frac{1}{4}\sum_{\text{spins}} 2\text{Re}\left(\mathcal{A}^{(0)*}\mathcal{A}^{(n)}\right)$$

- In the massless electron limit ($m_e^2 = 0$) 4-point process depending on 3 scales

- Integrand/integral reductions

$$\mathcal{M}^{(2)}(e\mu \to e\mu) = \sum_k c_k\left(s,t,m^2,\epsilon\right) I_k^{(2)}\left(s,t,m^2,\epsilon\right)$$

O(10000) monomials

O(100) MIs

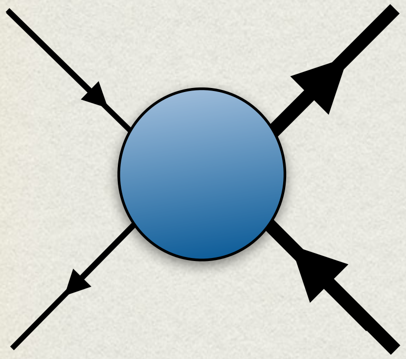
[Aida :: Mastrolia, Peraro, Primo, Ronca, W.J.T.]

+

[Reduze :: Studerus, von Manteuffel (2012)]

Algebraic decomposition

$$e^+e^- \to \mu^+\mu^-$$



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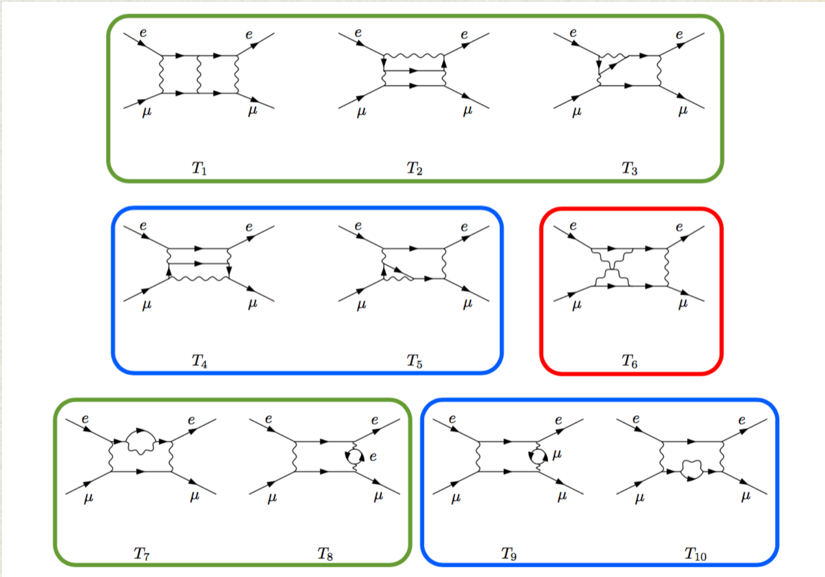
- Plug analytic expression of MIs

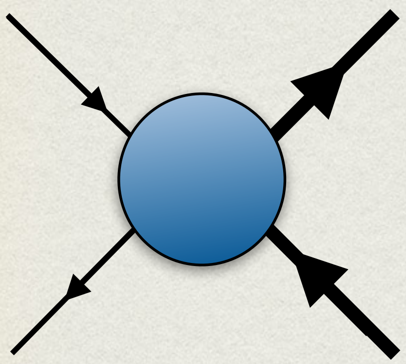
[Bonciani, Ferroglia, Gehrmann, von Manteuffel (2008-13)]
[Mastrolia, Passera, Primo, Schubert (2017)]
[Di Vita, Laporta, Mastrolia, Primo, Schubert (2018)]



Algebraic decomposition

$$\mathcal{A}^{(0)} = \mathcal{A}_{\rm b}^{(0)}$$
$$\mathcal{A}^{(1)} = \mathcal{A}_{\rm b}^{(1)} + \left(\delta Z_{\alpha}^{(1)} + \delta Z_{F}^{(1)}\right) \mathcal{A}_{\rm b}^{(0)}$$
$$\mathcal{A}^{(2)} = \mathcal{A}_{\rm b}^{(2)} + \left(2\delta Z_{\alpha}^{(1)} + \delta Z_{F}^{(1)}\right) \mathcal{A}_{\rm b}^{(1)} + \left(\delta Z_{\alpha}^{(2)} + \delta Z_{F}^{(2)} + \delta Z_{f}^{(2)} + \delta Z_{f}^{(1)} \delta Z_{\alpha}^{(1)}\right) \mathcal{A}_{\rm b}^{(0)} + \delta Z_{M}^{(1)} \mathcal{A}_{\rm b}^{(1,\text{mass CT})}$$

$$e^+e^- \to \mu^+\mu^-$$



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☑ **All bare amplitudes —> Computed!**

- In the massless electron limit ($m_e^2 = 0$) 4-point process depending on 3 scales

- Plug analytic expression of MIs

[Bonciani, Ferroglia, Gehrmann, von Manteuffel (2008-13)]
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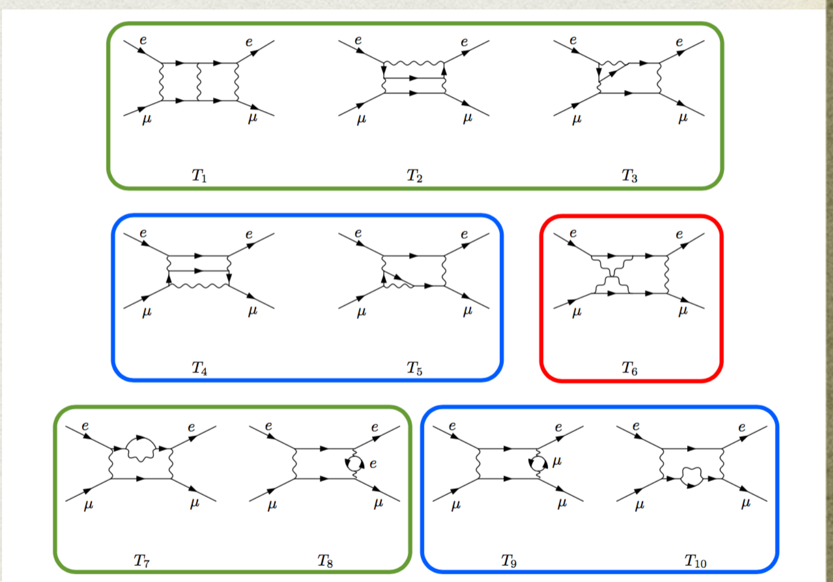


UV Renormalisation

- Fields

$$\psi_{\rm b} = \sqrt{Z_2}\psi\,, \quad A_{\rm b}^{\sigma} = \sqrt{Z_3}A^{\sigma}\,, \quad M_{\rm b} = Z_M M$$

- QED interaction vertex ⟶ Fixed from QED Ward id'

$$\mathcal{L}_{\rm int} = e_{\rm b}\,\bar{\psi}_{\rm b}\,\not{A}_{\rm b}\,\psi_{\rm b} = e\,\bar{\psi}\,\not{A}\,\psi$$

- Scheme :: On-shell + MSbar

$$Z_{2,e} = Z_{2,e}^{\rm OS}\,, \quad Z_{2,\mu} = Z_{2,\mu}^{\rm OS}\,, \quad Z_M = Z_M^{\rm OS}\,, \quad Z_{\alpha}^{\overline{\rm MS}} = 1/Z_3^{\overline{\rm MS}}$$

$$Z_j = 1 + \left(\frac{\alpha}{\pi}\right)\delta Z_j^{(1)} + \left(\frac{\alpha}{\pi}\right)^2 \delta Z_j^{(2)} + \mathcal{O}\left(\alpha^3\right)$$

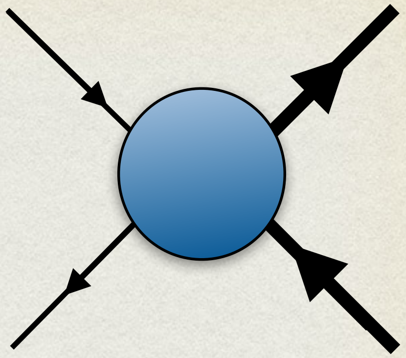
- UV Renormalised amplitudes

$$\mathcal{A}^{(0)} = \mathcal{A}_{\rm b}^{(0)}$$

$$\mathcal{A}^{(1)} = \mathcal{A}_{\rm b}^{(1)} + \left(\delta Z_{\alpha}^{(1)} + \delta Z_F^{(1)}\right)\mathcal{A}_{\rm b}^{(0)}$$

$$\mathcal{A}^{(2)} = \mathcal{A}_{\rm b}^{(2)} + \left(2\delta Z_{\alpha}^{(1)} + \delta Z_F^{(1)}\right)\mathcal{A}_{\rm b}^{(1)} + \left(\delta Z_{\alpha}^{(2)} + \delta Z_F^{(2)} + \delta Z_f^{(2)} + \delta Z_f^{(1)}\delta Z_{\alpha}^{(1)}\right)\mathcal{A}_{\rm b}^{(0)} + \delta Z_M^{(1)}\mathcal{A}_{\rm b}^{(1,\text{mass CT})}$$

Numerical evaluation of eμ @ two loops



$$\mathcal{M}^{(n)}\left(e^+e^- \to \mu^+\mu^-\right) = \frac{1}{4}\sum_{\text{spins}} 2\text{Re}\left(\mathcal{A}^{(0)*}\mathcal{A}^{(n)}\right)$$

$$\mathcal{M}^{(1)} = A^{(1)} + \boldsymbol{n_l}\, B_l^{(1)} + \boldsymbol{n_h}\, C_h^{(1)}$$

$$\mathcal{M}^{(2)} = A^{(2)} + \boldsymbol{n_l}\, B_l^{(2)} + \boldsymbol{n_h}\, C_h^{(2)} + \boldsymbol{n_l^2}\, D_l^{(2)} + \boldsymbol{n_h n_l}\, E_{hl}^{(2)} + \boldsymbol{n_h^2}\, F_h^{(2)}$$

- Evaluation @ $s/M^2 = 5\,, t/M^2 = -5/4\,, \mu = M.$

[Ginac :: Vollinga, Weinzierl (2004)]

[HandyG :: Naterop, Signer, Y. Ulrich (2019)]

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}	ϵ
$\mathcal{M}^{(0)}$	-	-	-	-	$\frac{181}{100}$	-2
$A^{(1)}$	-	-	$-\frac{181}{100}$	1.99877525	22.0079572	−11.7311017
$B_l^{(1)}$	-	-	-	-	−0.069056030	4.94328573
$C_h^{(1)}$	-	-	-	-	−2.24934027	2.54943566
$A^{(2)}$	$\frac{181}{400}$	−0.499387626	−35.4922919	19.4997261	48.8842283	-
$B_l^{(2)}$	-	$-\frac{181}{400}$	0.785712779	−16.1576674	−3.75247701	-
$C_h^{(2)}$	-	-	1.12467013	−9.50785825	−25.8771503	-
$D_l^{(2)}$	-	-	-	-	−3.96845688	-
$E_{hl}^{(2)}$	-	-	-	-	−4.88512563	-
$F_h^{(2)}$	-	-	-	-	−0.158490810	-

[Bonciani, Broggio, Di Vita, Ferroglia, Mandal, Mastrolia, Mattiazzi, Primo, Ronca, Schubert, W.J.T., Tramontano (2021)]

IR Structure w/ massive particles in the loop

- tree- & one-loop contributions —> two-loop IR poles after UV renormalisation

[Czakon, Mitov, Moch (2007)]
[Becher, Neubert (2009)]
[Hill (2017)]

$$\mathcal{M}^{(1)}\Big|_{\text{poles}} = \frac{1}{2} Z_1^{\text{IR}} \mathcal{M}^{(0)}\Big|_{\text{poles}}$$

$$\mathcal{M}^{(2)}\Big|_{\text{poles}} = \frac{1}{8} \left[\left(Z_2^{\text{IR}} - \left(Z_1^{\text{IR}}\right)^2 \right) \mathcal{M}^{(0)} + 2 Z_1^{\text{IR}} \mathcal{M}^{(1)} \right] \Big|_{\text{poles}}$$

- Anomalous dimension —> IR structure

$$\Gamma = \gamma_{\text{cusp}}(\alpha) \ln\left(-\frac{s}{\mu^2}\right) + 2\gamma_{\text{cusp}}(\alpha) \ln\left(\frac{t-M^2}{u-M^2}\right) + \gamma_{\text{cusp},M}(\alpha, s) + \gamma_h(\alpha, s) + \gamma_l(\alpha, s)$$

- IR renormalisation factor

$$\ln Z^{\text{IR}} = \frac{\alpha}{4\pi}\left(\frac{\Gamma_0'}{4\epsilon^2} + \frac{\Gamma_0}{2\epsilon}\right) + \left(\frac{\alpha}{4\pi}\right)^2 \left(-\frac{3\beta_0\Gamma_0'}{16\epsilon^3} + \frac{\Gamma_1' - 4\beta_0\Gamma_0}{16\epsilon^2} + \frac{\Gamma_1}{4\epsilon}\right) + \mathcal{O}\left(\alpha^3\right) \qquad \Gamma' = \frac{\partial}{\partial \ln\mu}\Gamma(\alpha)$$

☑ Full agreement of the IR poles structure obtained by direct calculation of the two-loop diagrams

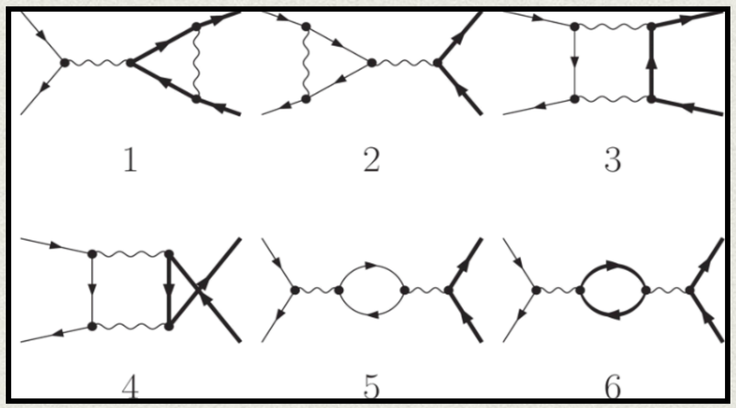
real & imaginary part!

$$e^+e^- \rightarrow \mu^+\mu^-$$

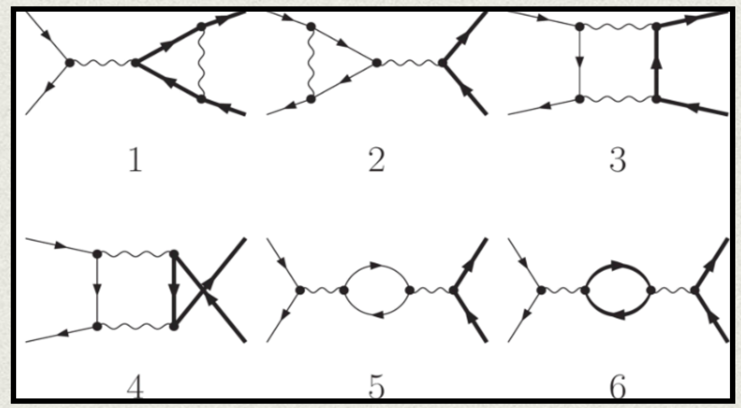
(one-loop) x (one-loop)

Anatomy of $e^+e^- \to \mu^+\mu^-$ (one-loop x one-loop)

- (One-loop) x (One-loop)



x



- Extensive use of tensor reduction :: FeynCalc
  - Product of one-loop master integrals

$$\sum_{ij} c_{ij}\left(s,t,M^2;\epsilon\right) I_i\left(s,t,M^2;\epsilon\right) I_i^*\left(s,t,M^2;\epsilon\right)$$

MIs needed up-to $\mathcal{O}\left(\epsilon^2\right)$

complex conjugate MIs

  - Read off all contributions

$$\mathcal{M}^{(1+1)} = \sum_{k=-4}^{0} \mathcal{M}_k^{(1+1)} \epsilon^k + \mathcal{O}(\varepsilon)$$

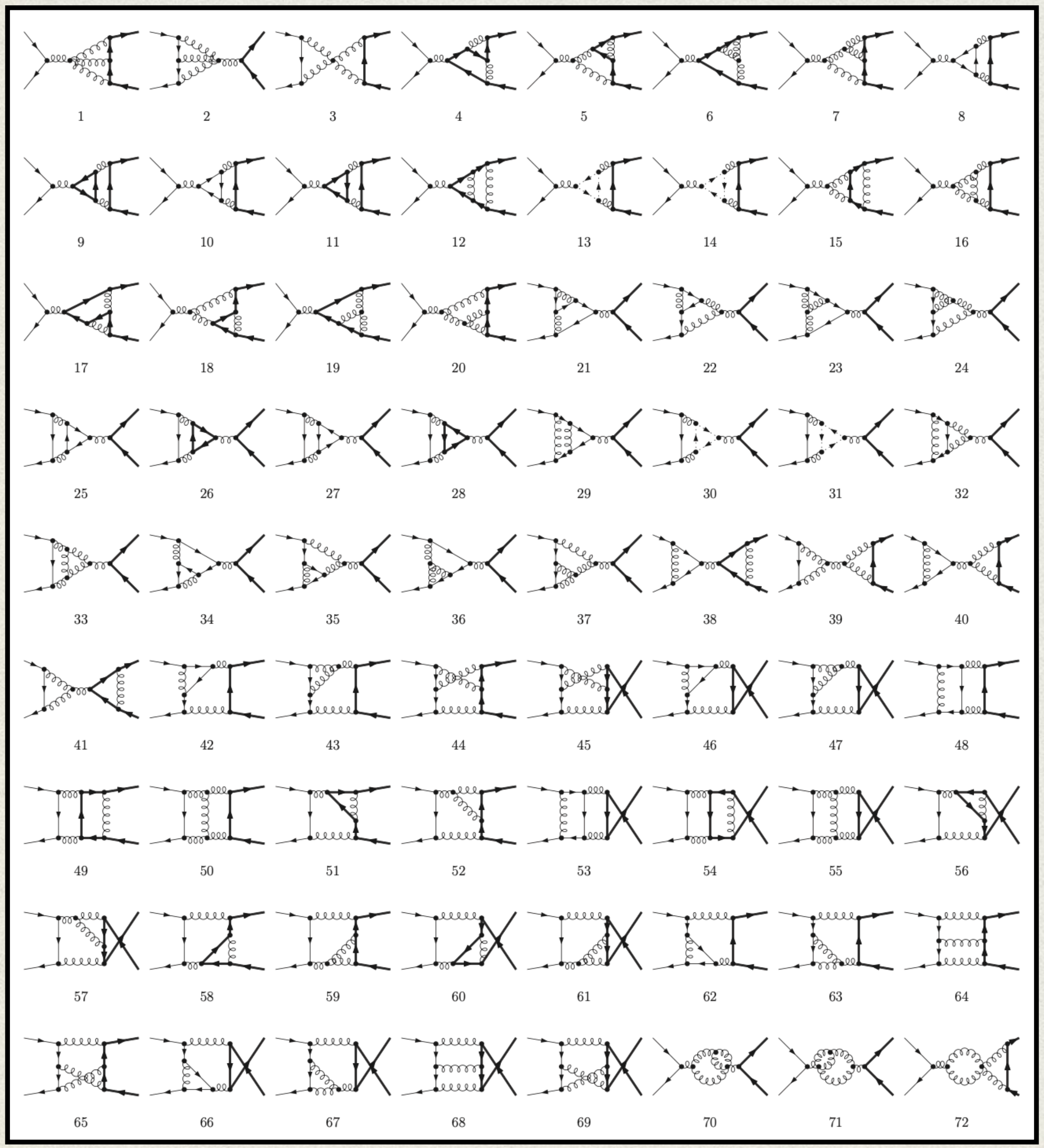
$q\bar{q} \to t\bar{t}$ @ two loops

Anatomy of $q\bar{q} \to t\bar{t}$ (up-to two loops)

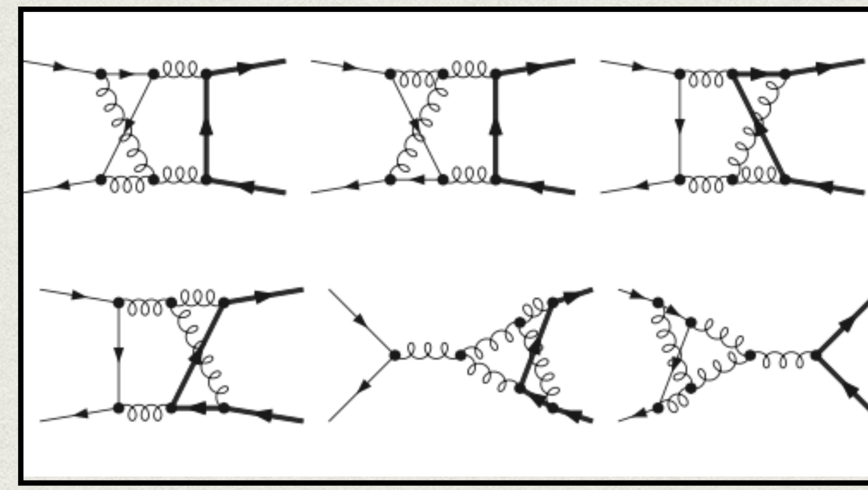
- Same kinematics as in $e^+e^- \to \mu^+\mu^-$

[Mandal, Mastrolia, Ronca, W.J.T. (2022)]

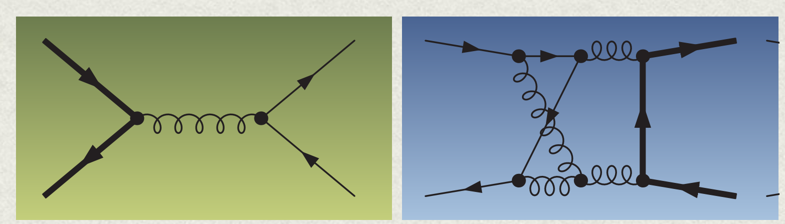
- Two-loop :: more diagrams ~220



- New topologies appear



- Vanish because of colour algebra



$$\propto T^a_{ij} T^a_{lk} f^{abc} \left(T^d T^c\right)_{kl} \left(T^b T^d T^a\right)_{ji} = 0$$

- No additional master integrals needed

Numerical evaluation $q\bar{q} \to t\bar{t}$ at two loops

[Mandal, Mastrolia, Ronca, W.J.T. (2022)]

$$
\mathcal{M}^{(2)} = 2\left(N_c^2 - 1\right)\left(A^{(2)}\, N_c^2 + B^{(2)} + \frac{C^{(2)}}{N_c^2} + D_l^{(2)}\, N_c\, n_l + D_h^{(2)}\, N_c\, n_h + E_l^{(2)}\, \frac{n_l}{N_c} + E_h^{(2)}\, \frac{n_h}{N_c} + F_l^{(2)}\, n_l^2 + F_{lh}^{(2)}\, n_l\, n_h + F_h^{(2)}\, n_h^2\right).
$$

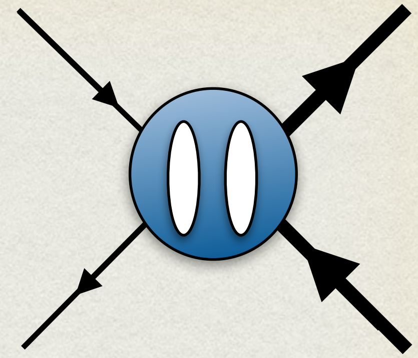
- Evaluation @ $s/M^2 = 5\,, t/M^2 = -5/4\,, \mu = M.$
- Full agreement w/ literature

	ϵ^{-4}	ϵ^{-3}	ϵ^{-2}	ϵ^{-1}	ϵ^{0}	ϵ^{1}
$A^{(0)}$	-	-	-	-	$\frac{181}{100}$	-2
$A^{(1)}$	-	-	$-\frac{181}{400}$	0.1026418456757775	1.356145770566065	2.230403451742140
$B^{(1)}$	-	-	$\frac{181}{400}$	-0.3180868339485723	-5.763132746701004	2.913169881363488
$C_l^{(1)}$	-	-	0	0	-0.01726400752682416	1.235821434465827
$C_h^{(1)}$	-	-	0	0	-0.5623350683773134	0.6373589172648111
$A^{(2)}$	$\frac{181}{800}$	1.391733154324222	-2.298174307221209	-4.145752448999165	17.37136598564062	-
$B^{(2)}$	$-\frac{181}{400}$	-1.323646320375650	8.507455541210568	6.035611156200398	-35.12861106350758	-
$C^{(2)}$	$\frac{181}{800}$	-0.06808683394857230	-18.00716652035224	6.302454931016090	3.524044912826756	-
$D_l^{(2)}$	0	- $\frac{181}{800}$	0.2605057338631945	-0.7250180282219092	-1.935417246635768	-
$D_h^{(2)}$	0	0	0.5623350683773134	0.1045606449242690	-1.704747997587188	-
$E_l^{(2)}$	0	$\frac{181}{800}$	-0.3323207299541260	7.904121951420471	2.848697836597635	-
$E_h^{(2)}$	0	0	-0.5623350683773134	4.528240788258799	12.73232424278180	-
$F_l^{(2)}$	0	0	0	0	-1.984228442234312	-
$F_{lh}^{(2)}$	0	0	0	0	-2.442562819239786	-
$F_h^{(2)}$	0	0	0	0	-0.07924540546146283	-

[Czakon(2008)]
[Bonciani, Ferroglia, Gehrmann, Maitre, Studerus (2008)]
[Bärnreuther, Czakon, Fiedler (2014)]

Results

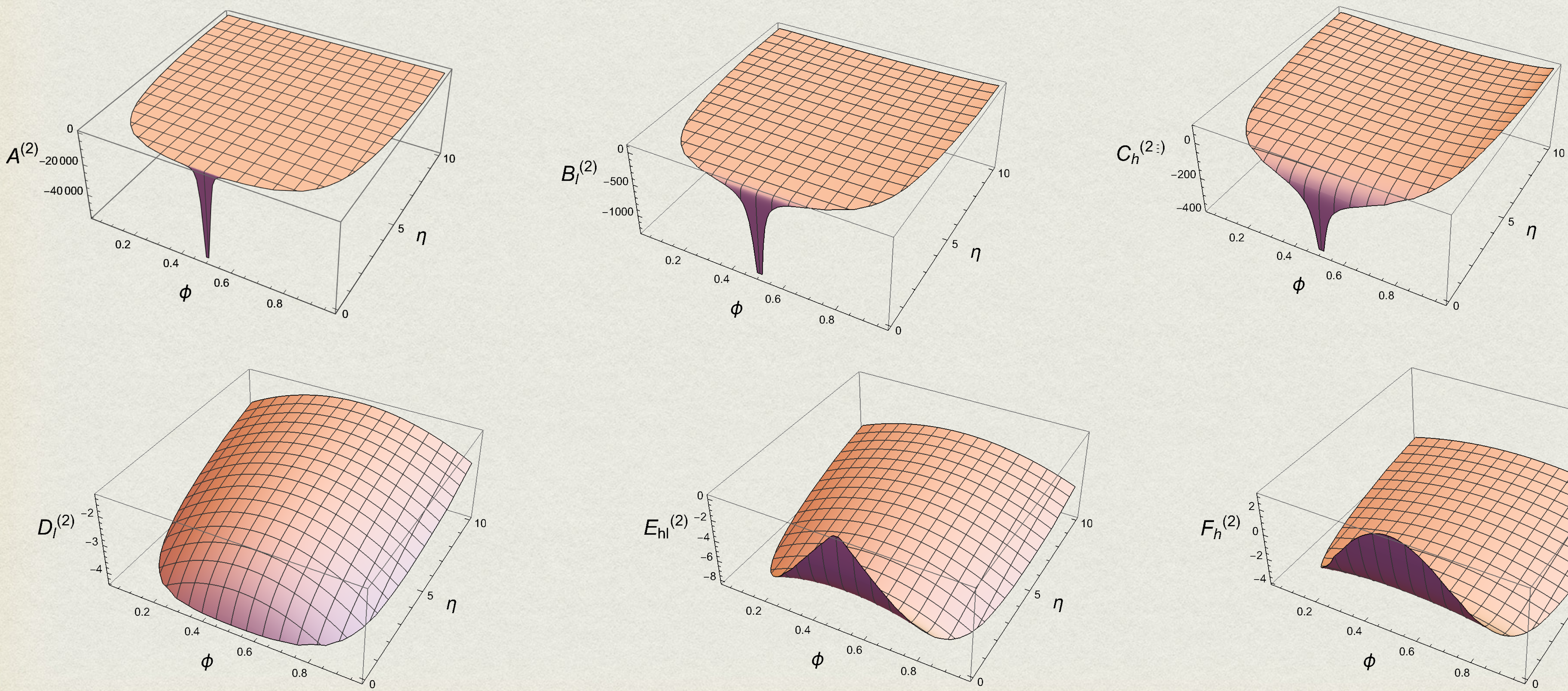
Finite reminders for $e^+e^- \to \mu^+\mu^-$ @ two-loop



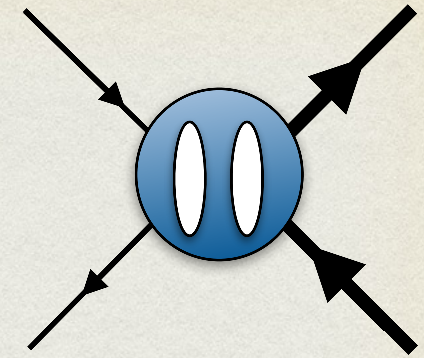
$$\mathcal{M}^{(2)}\left(e^+e^- \to \mu^+\mu^-\right) = \frac{1}{4}\sum_{\text{spins}} 2\text{Re}\left(\mathcal{A}^{(0)*}\mathcal{A}^{(2)}\right)$$

$$\mathcal{M}^{(2)} = A^{(2)} + \boldsymbol{n_l}\, B_l^{(2)} + \boldsymbol{n_h}\, C_h^{(2)} + \boldsymbol{n_l^2}\, D_l^{(2)} + \boldsymbol{n_h n_l}\, E_{hl}^{(2)} + \boldsymbol{n_h^2}\, F_h^{(2)}$$

$$\eta = \frac{s}{4M^2} - 1\,, \phi = -\frac{t-m^2}{s}\,,$$
$$\frac{1}{2}\left(1-\sqrt{\frac{\eta}{1+\eta}}\right) \le \phi \le \frac{1}{2}\left(1+\sqrt{\frac{\eta}{1+\eta}}\right)$$



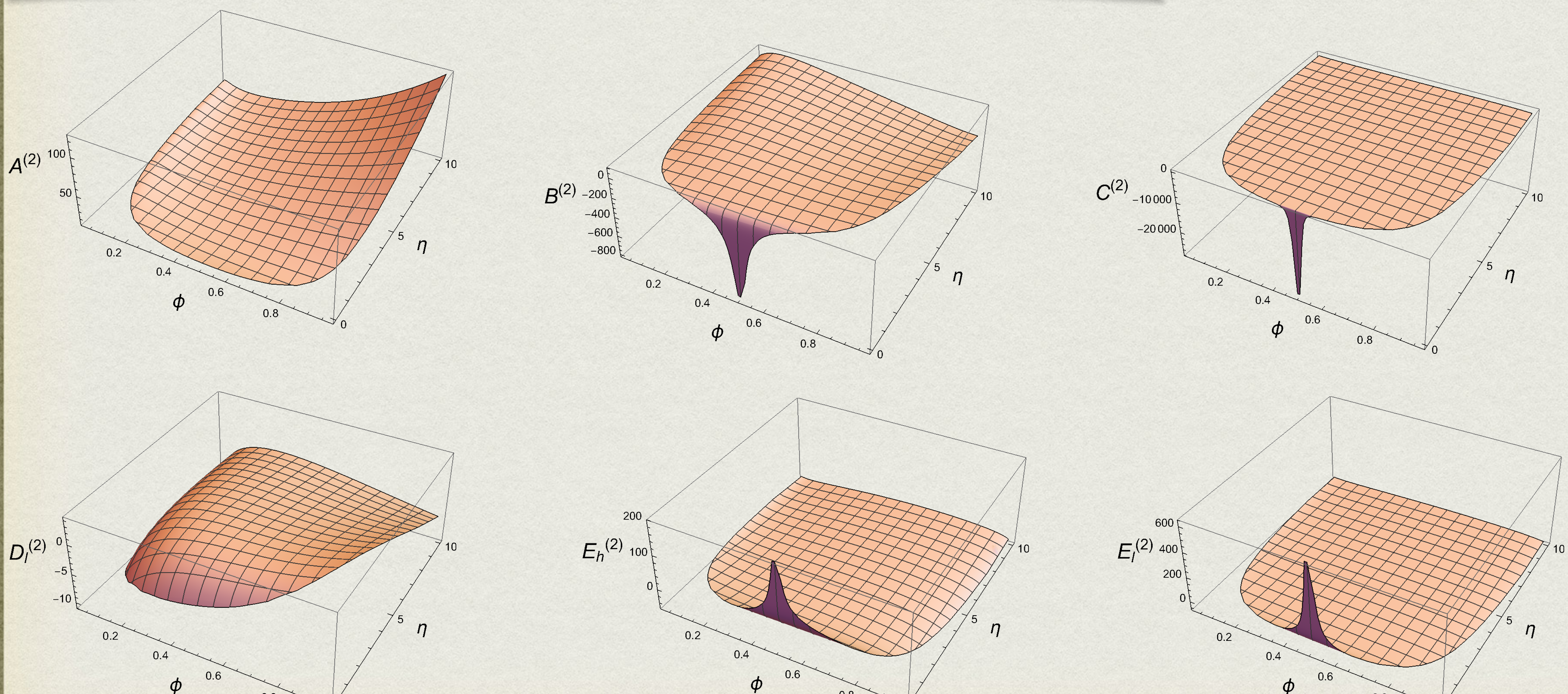
Finite reminders for $q\bar{q} \to t\bar{t}$ @ two-loop



$$\mathcal{M}^{(2)}\left(e^+e^- \to \mu^+\mu^-\right) = \frac{1}{4}\sum_{\text{spins}} 2\text{Re}\left(\mathcal{A}^{(0)*}\mathcal{A}^{(2)}\right)$$

$$\mathcal{M}^{(2)} = 2\left(N_c^2-1\right)\left(A^{(2)}N_c^2 + B^{(2)} + \frac{C^{(2)}}{N_c^2} + D_l^{(2)}N_c n_l + D_h^{(2)}N_c n_h + E_l^{(2)}\frac{n_l}{N_c} + E_h^{(2)}\frac{n_h}{N_c} + F_l^{(2)}n_l^2 + F_{lh}^{(2)}n_l n_h + F_h^{(2)}n_h^2\right).$$

$$\eta = \frac{s}{4M^2} - 1\,,\phi = -\frac{t-m^2}{s}\,,$$
$$\frac{1}{2}\left(1-\sqrt{\frac{\eta}{1+\eta}}\right) \le \phi \le \frac{1}{2}\left(1+\sqrt{\frac{\eta}{1+\eta}}\right)$$



Conclusions

Conclusions

- We have reached:

  - ☑ First QED analytical two-loop calculation for di-muon production process
  - ☑ Inclusion of non-zero electron mass to electron-muon elastic scattering calculation: massification
  - ☑ Differential and total cross section @ NNLO

- Open questions & future directions

  - ☐ Threshold expansion for both di-muon and top-pair production @NNLO
  - ☐ Extension to further QED processes (e.g. Compton scattering)
  - ☐ Towards the NNNLO frontier