The massless single off-shell scalar box integral

Branch cut structure and all-order ε-expansion

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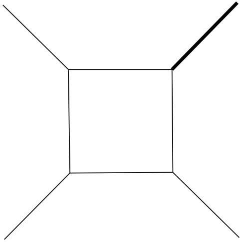
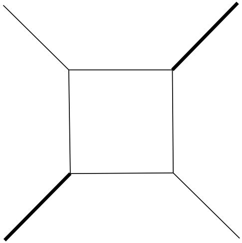
Universität Tübingen

Regensburg, February 2023





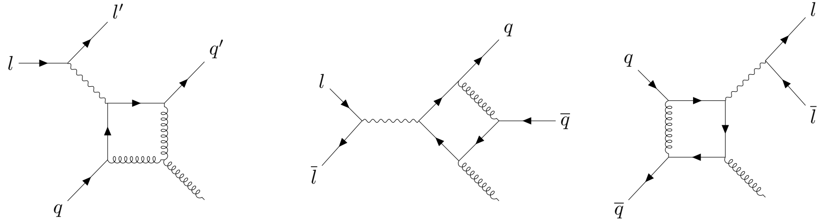
JH, Fabian Wunder, arXiv:2211.14110
JH, Fabian Wunder, arXiv:2302.01956

Outline

1. Motivation
2. Calculating the single off-shell scalar box integral
3. Generalization to two non-adjacent off-shell external particles
4. All-order ε-expansion of the scalar box integral
5. Conclusion and Outlook

1. Motivation

Motivation



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- Explicitly give real and imaginary parts

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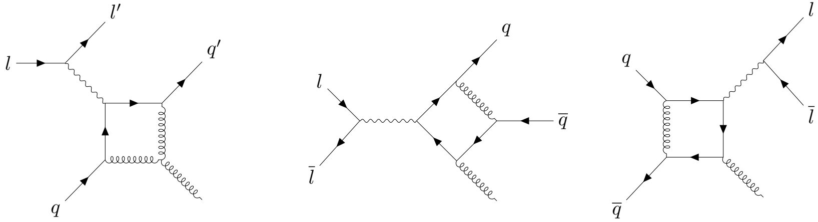

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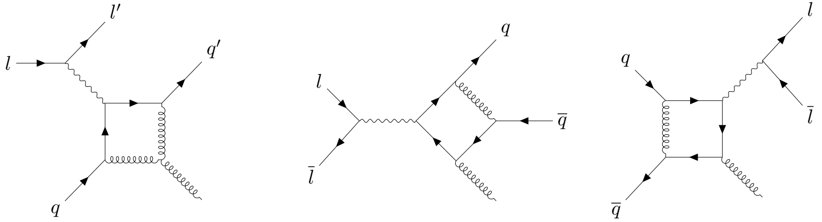

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The massless scalar box integral in the literature

Massless single-off shell case:

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Non-adjacent double off-shell case:

- Tarasov [2019]: Functional equation approach yields result in terms of 4 Gauss hypergeometric functions, expansion up to $\mathcal{O}(\varepsilon^0)$

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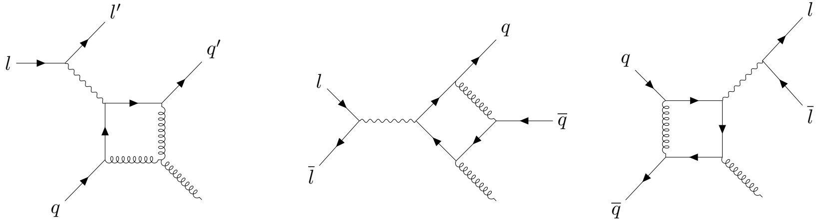

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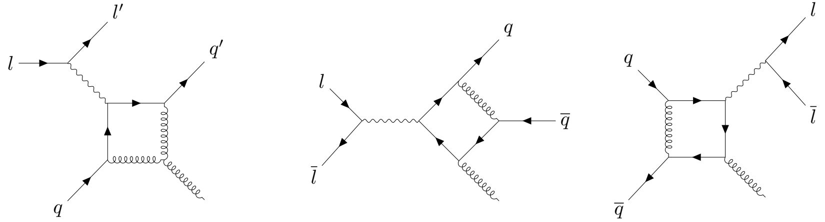

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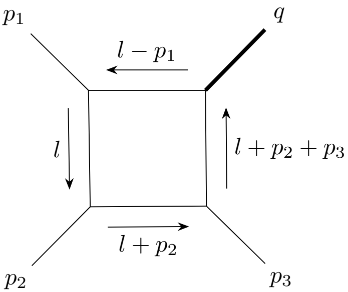
The scalar box integral in dimensional regularization



- External momenta taken to be incoming
- massless propagators
- Dimensional regularization with $d = 4 - 2\varepsilon$
- Keep causal $+\mathrm{i}0$ throughout
- $q^2 \neq 0,\ p_i^2 = 0$

$$\mathrm{D}_0 \equiv \frac{\mu^{4-d}}{\mathrm{i}\pi^{d/2}} \int \mathrm{d}^d l \, \frac{1}{[l^2 + \mathrm{i}0]\,[(l+p_2)^2 + \mathrm{i}0]\,[(l+p_2+p_3)^2 + \mathrm{i}0]\,[(l-p_1)^2 + \mathrm{i}0]}$$

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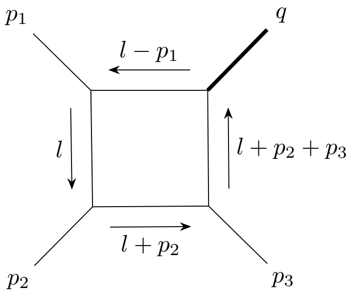

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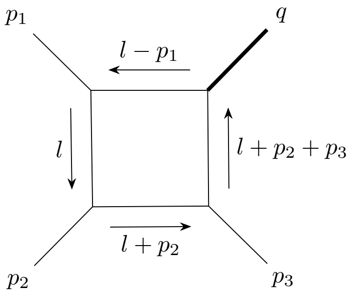

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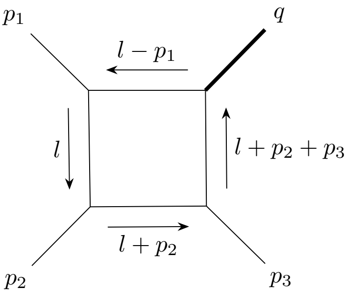

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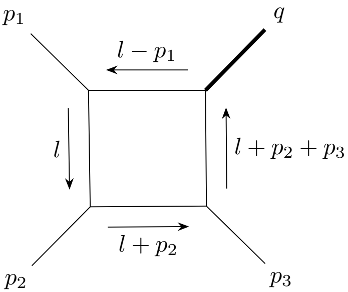

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Feynman parametrization

Feynman parametrization, evaluate loop integral →

$$D_0 = \mu^{2\varepsilon}\,\Gamma(2+\varepsilon)\int_0^1 \frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\,\delta(1-x_1-x_2-x_3-x_4)}{[-s_1x_1(x_2+x_3)-s_2x_3(x_1+x_4)-s_3x_1x_3-\mathrm{i}0]^{2+\varepsilon}}\,,$$

with Mandelstam variables

$$s_1 = (p_1+p_2)^2\,,\quad s_2 = (p_2+p_3)^2\,,\quad s_3 = (p_1+p_3)^2$$

Decouple Feynman parameter integrals through [Smirnov, 2012]

$$x_1 \to \eta_1\xi_1\,,\quad x_4 \to \eta_1(1-\xi_1)\,,$$
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Evaluate η-integrals in terms of Beta function →

$$D_0 = \mu^{2\varepsilon}\,\frac{\Gamma(2+\varepsilon)\Gamma^2(-\varepsilon)}{\Gamma(-2\varepsilon)}\int_0^1 \mathrm{d}\xi_1\int_0^1 \mathrm{d}\xi_2\,[-s_1\xi_1-s_2\xi_2-s_3\xi_1\xi_2-\mathrm{i}0]^{-\varepsilon-2}$$

ξ-integrals symmetric under $s_1 \leftrightarrow s_2$, function of 3 variables

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Factoring out $s_1 s_2 / s_3$

Use

$$(a - \mathrm{i}0)^\alpha = (b - \mathrm{i}0)^\alpha \left(\frac{a}{b} - \mathrm{i}0 \,\mathrm{sgn}(b) \right)^\alpha , \quad \text{where } a \in \mathbb{R} \,,\; b \in \mathbb{R}\backslash\{0\} \,,\; \alpha \in \mathbb{C} \,,$$

to factor out $s_1 s_2 / s_3 \rightarrow$

$$D_0\left(s_1, s_2, q^2\right) = \mu^{2\varepsilon} \frac{\Gamma(2+\varepsilon)\Gamma^2(-\varepsilon)}{\Gamma(-2\varepsilon)} \left(\frac{s_1 s_2}{s_3} - \mathrm{i}0 \right)^{-\varepsilon-2}$$
$$\times \int_0^1 \mathrm{d}\xi_1 \int_0^1 \mathrm{d}\xi_2 \left[x_2\xi_1 + x_1\xi_2 - x_1 x_2 \xi_1 \xi_2 - \mathrm{i}0 \,\mathrm{sgn}\left(\frac{s_3}{s_1 s_2} \right) \right]^{-\varepsilon-2} ,$$

depends on only 2 dimensionless variables

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Several substitutions & evaluating 1 integral & splitting of integrals $\rightarrow$

$$D_0\left(s_1, s_2, q^2\right) = -\frac{1}{\varepsilon} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left(\frac{s_3 \mu^2}{s_1 s_2} + \mathrm{i}0\right)^{\varepsilon} \times \left\{ I(x_1) + I(x_2) - I(1-(1-x_1)(1-x_2)) \right\}, \qquad (1)$$

where

$$I(\chi) \equiv \int_0^{\chi} \frac{\mathrm{d}\zeta}{1-\zeta} \left([\zeta - \mathrm{i}0\,\mathrm{sgn}_{123}]^{-\varepsilon-1} - 1 \right),$$

with abbreviation

$$\mathrm{sgn}_{123} \equiv \mathrm{sgn}\left(\frac{s_3}{s_1 s_2}\right)$$

Note

$$1-(1-x_1)(1-x_2) = -\frac{s_3 q^2}{s_1 s_2} \xrightarrow{q^2 \to 0} 0$$

$I(\chi)$ vanishes for $\chi = 0 \rightarrow$ set third integral to 0 in case $q^2 = 0$

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Calculating $I(\chi)$

In case $\chi \neq 0$, substitute $\zeta \to \chi^{-1}\zeta \to$

$$I(\chi) = \int_0^1 \frac{\mathrm{d}\zeta}{1-\chi\zeta}\left([\chi - \mathrm{i}0\,\mathrm{sgn}_{123}]^{-\varepsilon}\,\zeta^{-\varepsilon-1} - \chi\right)$$

Denominator $1-\chi\zeta$ diverges for $\chi > 1$

$\to$ Introduce regulator $\chi \to \chi + \mathrm{i}\tilde{0}$ to split integral in two

$$\begin{aligned} I(\chi) &= [\chi - \mathrm{i}0\,\mathrm{sgn}_{123}]^{-\varepsilon}\int_0^1 \mathrm{d}\zeta\,\zeta^{-\varepsilon-1}\left(1-(\chi+\mathrm{i}\tilde{0})\zeta\right)^{-1} - \int_0^1 \mathrm{d}\zeta\,\frac{\chi}{1-(\chi+\mathrm{i}\tilde{0})\zeta} \\ &= -\frac{1}{\varepsilon}[\chi - \mathrm{i}0\,\mathrm{sgn}_{123}]^{-\varepsilon}\,{}_2F_1(1,-\varepsilon,1-\varepsilon;\chi+\mathrm{i}\tilde{0}) + \ln(1-\chi-\mathrm{i}\tilde{0}) \end{aligned}$$

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Calculating $I(\chi)$

In case $\chi \neq 0$, substitute $\zeta \to \chi^{-1}\zeta \to$

$$I(\chi) = \int_0^1 \frac{\mathrm{d}\zeta}{1-\chi\zeta} \left([\chi - \mathrm{i}0\,\mathrm{sgn}_{123}]^{-\varepsilon} \zeta^{-\varepsilon-1} - \chi \right)$$

Denominator $1-\chi\zeta$ diverges for $\chi > 1$

$\to$ Introduce regulator $\chi \to \chi + \mathrm{i}\tilde{0}$ to split integral in two

$$\begin{aligned} I(\chi) &= [\chi - \mathrm{i}0\,\mathrm{sgn}_{123}]^{-\varepsilon} \int_0^1 \mathrm{d}\zeta\, \zeta^{-\varepsilon-1} \left(1-(\chi+\mathrm{i}\tilde{0})\zeta\right)^{-1} - \int_0^1 \mathrm{d}\zeta\, \frac{\chi}{1-(\chi+\mathrm{i}\tilde{0})\zeta} \\ &= -\frac{1}{\varepsilon} [\chi - \mathrm{i}0\,\mathrm{sgn}_{123}]^{-\varepsilon}\, {}_2F_1\left(1,-\varepsilon,1-\varepsilon;\chi+\mathrm{i}\tilde{0}\right) + \ln(1-\chi-\mathrm{i}\tilde{0}) \end{aligned}$$

- Both ${}_2F_1$ and ln evaluated on branch cut for $\chi > 1$
- Branch cuts are spurious and must cancel

Cancellation of spurious branch cuts

Add integrals in eq. (1),

$$D_0\left(s_1, s_2, q^2\right) = -\frac{1}{\varepsilon}\frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\,\frac{2}{s_1 s_2}\left(\frac{s_3\mu^2}{s_1 s_2}+\mathrm{i}0\right)^{\varepsilon} \times \left\{I(x_1) + I(x_2) - I(1-(1-x_1)(1-x_2))\right\},$$

use different regulator $\mathrm{i}\tilde{0}_i$ for each integral

Sum $I(x_1) + I(x_2) - I(1-(1-x_1)(1-x_2))$ contains following logarithms

$$\ln(1-x_1-\mathrm{i}\tilde{0}_1) + \ln(1-x_2-\mathrm{i}\tilde{0}_2) - \ln\big((1-x_1)(1-x_2)-\mathrm{i}\tilde{0}_3\big)$$

- Real parts cancel
- Choose signs of $\mathrm{i}\tilde{0}_i$ such that imaginary parts cancel as well

Cancellation of spurious branch cuts

Add integrals in eq. (1),

$$D_0\left(s_1, s_2, q^2\right) = -\frac{1}{\varepsilon}\frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\,\frac{2}{s_1 s_2}\left(\frac{s_3\mu^2}{s_1 s_2} + \mathrm{i}0\right)^{\varepsilon} \times \left\{I(x_1) + I(x_2) - I(1-(1-x_1)(1-x_2))\right\},$$

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Cancellation of spurious branch cuts

Add integrals in eq. (1),

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Sum $I(x_1) + I(x_2) - I(1-(1-x_1)(1-x_2))$ contains following logarithms

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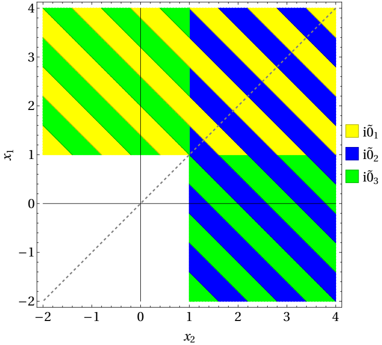
- Real parts cancel
- Choose signs of $\mathrm{i}\tilde{0}_i$ such that imaginary parts cancel as well

Cancellation of imaginary parts

$$i\pi \left[-\operatorname{sgn}\left(\tilde{0}_1\right) \Theta(x_1 - 1) - \operatorname{sgn}\left(\tilde{0}_2\right) \Theta(x_2 - 1) + \operatorname{sgn}\left(\tilde{0}_3\right) \left\{ \Theta(x_1 - 1)\,\Theta(1 - x_2) + \Theta(1 - x_1)\,\Theta(x_2 - 1) \right\} \right] \overset{!}{=} 0 \qquad (2)$$



- Conditions for eq. (2) to hold:

$$\operatorname{sgn}\left(\tilde{0}_1\right) \overset{!}{=} \operatorname{sgn}\left(\tilde{0}_3\right) \quad \text{(yellow-green region)}$$

$$\operatorname{sgn}\left(\tilde{0}_1\right) \overset{!}{=} -\operatorname{sgn}\left(\tilde{0}_2\right) \quad \text{(yellow-blue region)}$$

$$\operatorname{sgn}\left(\tilde{0}_2\right) \overset{!}{=} \operatorname{sgn}\left(\tilde{0}_3\right) \quad \text{(blue-green region)}$$

- Choose (only relative signs matter)

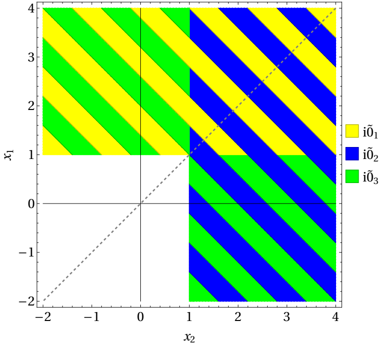
$$i\tilde{0}_1 \equiv i\tilde{0}\operatorname{sgn}(x_1 - x_2)$$

$$i\tilde{0}_2 \equiv i\tilde{0}\operatorname{sgn}(x_2 - x_1)$$

$$i\tilde{0}_3 \equiv i\tilde{0}$$

Cancellation of imaginary parts

$$i\pi \left[-\operatorname{sgn}(\tilde{0}_1)\,\Theta(x_1-1) - \operatorname{sgn}(\tilde{0}_2)\,\Theta(x_2-1) + \operatorname{sgn}(\tilde{0}_3)\,\{\Theta(x_1-1)\,\Theta(1-x_2) + \Theta(1-x_1)\,\Theta(x_2-1)\}\right] \stackrel{!}{=} 0 \qquad (2)$$



- Conditions for eq. (2) to hold:

  $\operatorname{sgn}(\tilde{0}_1) \stackrel{!}{=} \operatorname{sgn}(\tilde{0}_3)$ (yellow-green region)

  $\operatorname{sgn}(\tilde{0}_1) \stackrel{!}{=} -\operatorname{sgn}(\tilde{0}_2)$ (yellow-blue region)

  $\operatorname{sgn}(\tilde{0}_2) \stackrel{!}{=} \operatorname{sgn}(\tilde{0}_3)$ (blue-green region)

- Choose (only relative signs matter)

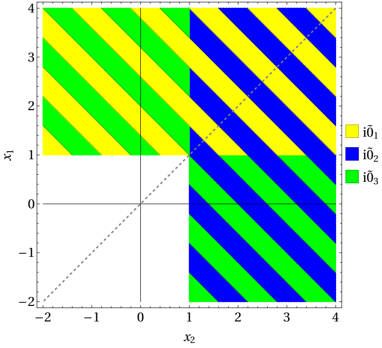
  $i\tilde{0}_1 \equiv i\tilde{0}\,\operatorname{sgn}(x_1-x_2)$

  $i\tilde{0}_2 \equiv i\tilde{0}\,\operatorname{sgn}(x_2-x_1)$

  $i\tilde{0}_3 \equiv i\tilde{0}$

Cancellation of imaginary parts

$$i\pi\left[-\operatorname{sgn}(\tilde{0}_1)\,\Theta(x_1-1)-\operatorname{sgn}(\tilde{0}_2)\,\Theta(x_2-1)\right.$$
$$\left.+\operatorname{sgn}(\tilde{0}_3)\,\{\Theta(x_1-1)\,\Theta(1-x_2)+\Theta(1-x_1)\,\Theta(x_2-1)\}\right]\stackrel{!}{=}0 \qquad (2)$$



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$$\operatorname{sgn}(\tilde{0}_2)\stackrel{!}{=}\operatorname{sgn}(\tilde{0}_3)\quad\text{(blue-green region)}$$

- Choose (only relative signs matter)

$$i\tilde{0}_1 \equiv i\tilde{0}\,\operatorname{sgn}(x_1-x_2)$$
$$i\tilde{0}_2 \equiv i\tilde{0}\,\operatorname{sgn}(x_2-x_1)$$
$$i\tilde{0}_3 \equiv i\tilde{0}$$

Result in terms of hypergeometric functions

$$
\begin{aligned}
D_0\left(s_1, s_2, q^2\right) = \; & \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left(\frac{s_3 \mu^2}{s_1 s_2} + \mathrm{i}0\right)^{\varepsilon} \\
& \times \left\{ \left[-\frac{s_3}{s_1} - \mathrm{i}0 \,\mathrm{sgn}\left(\frac{s_3}{s_1 s_2}\right)\right]^{-\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon;\; -\frac{s_3}{s_1} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_2} - \frac{s_3}{s_1}\right)\right) \right. \\
& \quad + \left[-\frac{s_3}{s_2} - \mathrm{i}0 \,\mathrm{sgn}\left(\frac{s_3}{s_1 s_2}\right)\right]^{-\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon;\; -\frac{s_3}{s_2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_1} - \frac{s_3}{s_2}\right)\right) \\
& \quad \left. - \left[-\frac{s_3 q^2}{s_1 s_2} - \mathrm{i}0 \,\mathrm{sgn}\left(\frac{s_3}{s_1 s_2}\right)\right]^{-\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon;\; -\frac{s_3 q^2}{s_1 s_2} + \mathrm{i}\tilde{0}\right) \right\} \qquad (3)
\end{aligned}
$$

- Imaginary parts of hypergeometric functions will cancel by construction
- Last term vanishes for $q^2 = 0$

Result in terms of hypergeometric functions

$$
\begin{aligned}
D_0\left(s_1, s_2, q^2\right) = &\ \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left(\frac{s_3 \mu^2}{s_1 s_2} + \mathrm{i}0\right)^{\varepsilon} \\
& \times \left\{ \left[-\frac{s_3}{s_1} - \mathrm{i}0\,\mathrm{sgn}\left(\frac{s_3}{s_1 s_2}\right)\right]^{-\varepsilon} {}_2\mathrm{F}_1\left(1, -\varepsilon, 1-\varepsilon;\ -\frac{s_3}{s_1} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_2} - \frac{s_3}{s_1}\right)\right)\right. \\
& + \left[-\frac{s_3}{s_2} - \mathrm{i}0\,\mathrm{sgn}\left(\frac{s_3}{s_1 s_2}\right)\right]^{-\varepsilon} {}_2\mathrm{F}_1\left(1, -\varepsilon, 1-\varepsilon;\ -\frac{s_3}{s_2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_1} - \frac{s_3}{s_2}\right)\right) \\
& \left. - \left[-\frac{s_3 q^2}{s_1 s_2} - \mathrm{i}0\,\mathrm{sgn}\left(\frac{s_3}{s_1 s_2}\right)\right]^{-\varepsilon} {}_2\mathrm{F}_1\left(1, -\varepsilon, 1-\varepsilon;\ -\frac{s_3 q^2}{s_1 s_2} + \mathrm{i}\tilde{0}\right)\right\} \qquad (3)
\end{aligned}
$$

- Imaginary parts of hypergeometric functions will cancel by construction
- Last term vanishes for $q^2 = 0$

Comparison to literature

Combine prefactors $(\ldots)^\varepsilon$ and $[\ldots]^{-\varepsilon} \rightarrow$

$$D_0\left(s_1, s_2, q^2\right) = \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \\ \times \left\{ \left[\frac{\mu^2}{-s_2 - i0}\right]^\varepsilon {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3}{s_1} + i\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_2} - \frac{s_3}{s_1}\right)\right) \right. \\ + \left[\frac{\mu^2}{-s_1 - i0}\right]^\varepsilon {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3}{s_2} + i\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_1} - \frac{s_3}{s_2}\right)\right) \\ \left. - \left[\frac{\mu^2}{-q^2 - i0}\right]^\varepsilon {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3 q^2}{s_1 s_2} + i\tilde{0}\right) \right\}$$

- Agrees with [Matsuura et al., 1989] and [Lyubovitskij et al., 2021] if $s_1, s_2, q^2 < 0$ and all three hypergeometric functions are away from their branch cut
- Agrees with [Bern et al., 1994] all three hypergeometric functions are away from their branch cut

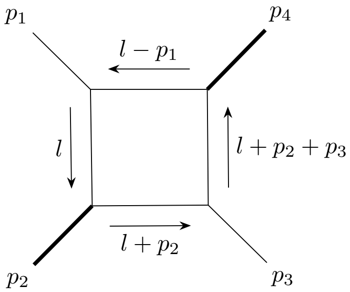
Comparison to literature

Combine prefactors $(\ldots)^{\varepsilon}$ and $[\ldots]^{-\varepsilon} \rightarrow$

$$
\begin{aligned}
D_0\left(s_1, s_2, q^2\right) = & \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \\
& \times \left\{ \left[\frac{\mu^2}{-s_2 - \mathrm{i}0}\right]^{\varepsilon} {}_2\mathrm{F}_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3}{s_1} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_2} - \frac{s_3}{s_1}\right)\right)\right. \\
& \quad + \left[\frac{\mu^2}{-s_1 - \mathrm{i}0}\right]^{\varepsilon} {}_2\mathrm{F}_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3}{s_2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_1} - \frac{s_3}{s_2}\right)\right) \\
& \quad \left. - \left[\frac{\mu^2}{-q^2 - \mathrm{i}0}\right]^{\varepsilon} {}_2\mathrm{F}_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{s_3 q^2}{s_1 s_2} + \mathrm{i}\tilde{0}\right)\right\}
\end{aligned}
$$

- Agrees with [Matsuura et al., 1989] and [Lyubovitskij et al., 2021] if $s_1, s_2, q^2 < 0$ and all three hypergeometric functions are away from their branch cut
- Agrees with [Bern et al., 1994] all three hypergeometric functions are away from their branch cut

3. Generalization to two non-adjacent off-shell external particles



JH, Fabian Wunder, arXiv:2302.01956

Generalization to two non-adjacent off-shell external particles

General box integral with massless propagators after Feynman parametrization:

$$D_0 = \mu^{2\varepsilon}\,\Gamma(2+\varepsilon) \times \int_0^1 \frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\,\delta(1-x_1-x_2-x_3-x_4)}{[-x_1x_2s_1 - x_1x_3p_4^2 - x_1x_4p_1^2 - x_2x_3p_3^2 - x_2x_4p_2^2 - x_3x_4s_2 - \mathrm{i}0]^{2+\varepsilon}}$$

With the same substitution as before,

$$x_1 = \eta_1\xi_1\,, \quad x_2 = \eta_2(1-\xi_2)\,, \quad x_3 = \eta_2\xi_2\,, \quad x_4 = \eta_1(1-\xi_1)\,,$$

term in denominator becomes

$$-\eta_1\eta_2(1-\xi_1)(1-\xi_2)\,s_1 - \eta_1\eta_2\xi_1\xi_2\,s_2 - \eta_1\eta_2\xi_1(1-\xi_2)\,p_2^2 - \eta_1\eta_2\xi_2(1-\xi_1)\,p_4^2 - \eta_1^2\xi_1(1-\xi_1)\,p_1^2 - \eta_2^2\xi_2(1-\xi_2)\,p_3^2 - \mathrm{i}0$$

- Factorization of η- and ξ-integrals if $p_1^2 = p_3^2 = 0$
- Proceed analogously to single off-shell case for two non-adjacent external particles off light cone

Generalization to two non-adjacent off-shell external particles

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$$D_0 = \mu^{2\varepsilon}\,\Gamma(2+\varepsilon) \times \int_0^1 \frac{\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\mathrm{d}x_4\,\delta(1-x_1-x_2-x_3-x_4)}{\left[-x_1x_2s_1 - x_1x_3p_4^2 - x_1x_4p_1^2 - x_2x_3p_3^2 - x_2x_4p_2^2 - x_3x_4s_2 - \mathrm{i}0\right]^{2+\varepsilon}}$$

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term in denominator becomes

$$-\eta_1\eta_2(1-\xi_1)(1-\xi_2)\,s_1 - \eta_1\eta_2\xi_1\xi_2\,s_2 - \eta_1\eta_2\xi_1(1-\xi_2)\,p_2^2 - \eta_1\eta_2\xi_2(1-\xi_1)\,p_4^2$$
$$- \eta_1^2\xi_1(1-\xi_1)\,p_1^2 - \eta_2^2\xi_2(1-\xi_2)\,p_3^2 - \mathrm{i}0$$

- Factorization of η- and ξ-integrals if $p_1^2 = p_3^2 = 0$
- Proceed analogously to single off-shell case for two non-adjacent external particles off light cone

Generalization to two non-adjacent off-shell external particles

General box integral with massless propagators after Feynman parametrization:

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term in denominator becomes

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- Factorization of η- and ξ-integrals if $p_1^2 = p_3^2 = 0$
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term in denominator becomes

$$-\eta_1\eta_2(1-\xi_1)(1-\xi_2)\,s_1 - \eta_1\eta_2\xi_1\xi_2\,s_2 - \eta_1\eta_2\xi_1(1-\xi_2)\,p_2^2 - \eta_1\eta_2\xi_2(1-\xi_1)\,p_4^2 - \eta_1^2\xi_1(1-\xi_1)\,p_1^2 - \eta_2^2\xi_2(1-\xi_2)\,p_3^2 - \mathrm{i}0$$

- Factorization of η- and ξ-integrals if $p_1^2 = p_3^2 = 0$
- Proceed analogously to single off-shell case for two non-adjacent external particles off light cone

Result in terms of 4 hypergeometric functions

$$
\begin{aligned}
D_0\left(s_1, s_2, 0, p_2^2, 0, p_4^2\right) = {}& \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2 - p_2^2 p_4^2} \\
\times \Bigg\{ & \left[\frac{\mu^2}{-s_1 - \mathrm{i}0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_1}{s_1 s_2 - p_2^2 p_4^2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}(s_1 - s_2)\right) \\
+ & \left[\frac{\mu^2}{-s_2 - \mathrm{i}0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_2}{s_1 s_2 - p_2^2 p_4^2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}(s_2 - s_1)\right) \\
- & \left[\frac{\mu^2}{-p_2^2 - \mathrm{i}0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)p_2^2}{s_1 s_2 - p_2^2 p_4^2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(p_4^2 - p_2^2\right)\right) \\
- & \left[\frac{\mu^2}{-p_4^2 - \mathrm{i}0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)p_4^2}{s_1 s_2 - p_2^2 p_4^2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(p_2^2 - p_4^2\right)\right) \Bigg\}
\end{aligned}
$$

- Reduces to single off-shell integral in limits $p_2^2, p_4^2 \to 0$
- Symmetric under simultaneously interchanging $s_1 \leftrightarrow s_2$ and $p_2^2 \leftrightarrow p_4^2$, as well as under $s_1 \leftrightarrow p_2^2$, $s_2 \leftrightarrow p_4^2$
- Similar result found via functional equations in [Tarasov, 2019]

Result in terms of 4 hypergeometric functions

$$D_0\left(s_1, s_2, 0, p_2^2, 0, p_4^2\right) = \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2 - p_2^2 p_4^2}$$
$$\times \left\{ \left[\frac{\mu^2}{-s_1 - \mathrm{i}0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_1}{s_1 s_2 - p_2^2 p_4^2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}(s_1 - s_2)\right)\right.$$
$$+ \left[\frac{\mu^2}{-s_2 - \mathrm{i}0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_2}{s_1 s_2 - p_2^2 p_4^2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}(s_2 - s_1)\right)$$
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- Reduces to single off-shell integral in limits $p_2^2, p_4^2 \to 0$
- Symmetric under simultaneously interchanging $s_1 \leftrightarrow s_2$ and $p_2^2 \leftrightarrow p_4^2$, as well as under $s_1 \leftrightarrow p_2^2$, $s_2 \leftrightarrow p_4^2$
- Similar result found via functional equations in [Tarasov, 2019]

Result in terms of 4 hypergeometric functions

$$
\begin{aligned}
D_0\left(s_1, s_2, 0, p_2^2, 0, p_4^2\right) = {} & \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2 - p_2^2 p_4^2} \\
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+ & \left[\frac{\mu^2}{-s_2 - \mathrm{i}0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_2}{s_1 s_2 - p_2^2 p_4^2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}(s_2 - s_1)\right) \\
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- & \left[\frac{\mu^2}{-p_4^2 - \mathrm{i}0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)p_4^2}{s_1 s_2 - p_2^2 p_4^2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(p_2^2 - p_4^2\right)\right) \Bigg\}
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4. All-order ε-expansion of the scalar box integral

Our ε-expansion

Write eq. (3) as

$$D_0\left(s_1, s_2, q^2\right) = \frac{1}{\varepsilon^2}\frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\,\frac{2}{s_1 s_2}\left|\frac{s_3\mu^2}{s_1 s_2}\right|^{\varepsilon} \times \exp\left[\mathrm{i}\pi\varepsilon\,\Theta\left(-\frac{s_3}{s_1 s_2}\right)\right]\mathcal{D}_0\left(\varepsilon; -\frac{s_3}{s_1}, -\frac{s_3}{s_2}\right),$$

where $\mathcal{D}_0$ abbreviates the sum of hypergeometric functions (upper sign choice for $x_1 \geq x_2$, else lower sign),

$$\mathcal{D}_0(\varepsilon; x_1, x_2) = \mathcal{F}_{\pm}(\varepsilon; x_1) + \mathcal{F}_{\mp}(\varepsilon; x_2) - \mathcal{F}_{+}(\varepsilon; 1-(1-x_1)(1-x_2))$$

Here,

$$\mathcal{F}_{\pm}(\varepsilon; x) \equiv (x - \mathrm{i}0\,\mathrm{sgn}_{123})^{-\varepsilon}\,{}_2\mathrm{F}_1\left(1, -\varepsilon, 1-\varepsilon; x \pm \mathrm{i}\tilde{0}\right),$$

Epsilon expansion of $\mathcal{F}_\pm(\varepsilon; x)$

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- Ingredients for expansion:
  Epsilon expansion of ${}_2\mathrm{F}_1$ [Lyubovitskij et al., 2021]

$$ {}_2\mathrm{F}_1(1, -\varepsilon, 1-\varepsilon; z) = 1 - \sum_{n=1}^{\infty} \varepsilon^n \,\mathrm{Li}_n(z)$$

  Inversion formula for ${}_2\mathrm{F}_1$ (use for $x > 1$)

$$ {}_2\mathrm{F}_1(1, -\varepsilon, 1-\varepsilon; z \pm \mathrm{i}\tilde{0}) + {}_2\mathrm{F}_1\left(1, \varepsilon, 1+\varepsilon; \frac{1}{z}\right) = 1 + (-z \mp \mathrm{i}\tilde{0})^{\varepsilon}\,\frac{\pi\varepsilon}{\sin(\pi\varepsilon)}$$

- Goal: Make cancellation of spurious branch cuts explicit

Epsilon expansion of $\mathcal{F}_{\pm}(\varepsilon; x)$

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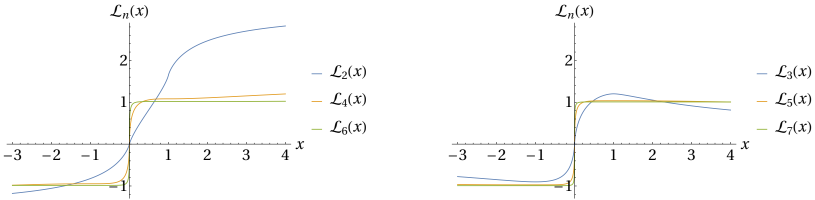
Single-valued polylogarithms

$$\mathcal{L}_n(x) \equiv \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \ln^k |x| \, \mathrm{Li}_{n-k}(x) + \frac{(-1)^{n-1}}{n!} \ln^{n-1} |x| \ln |1-x|$$



- $\mathcal{L}_n(x)$ is single-valued, in contrast to $\mathrm{Li}_n(x)$
- $\mathcal{L}_n(x)$ is continuous for all $x \in \mathbb{R}$
- $\mathcal{L}_n(x)$ is bounded on $\mathbb{R}$, in contrast to $\mathrm{Li}_n(x)$
- $\mathcal{L}_n(x)$ satisfies *clean* versions of the functional equations of $\mathrm{Li}_n(x)$, i.e. without product terms

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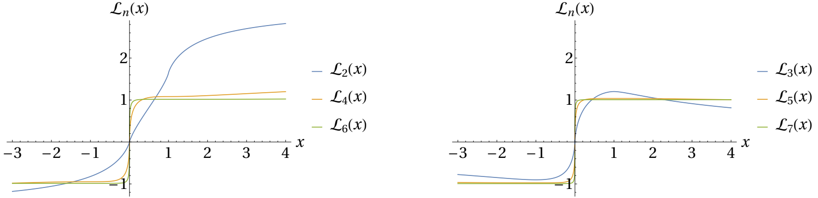

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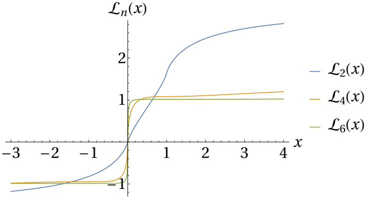

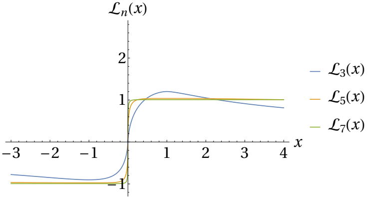

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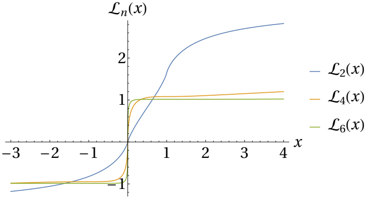

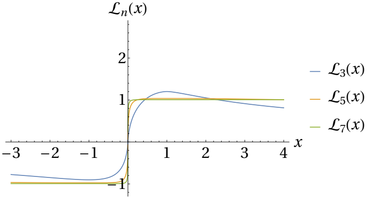

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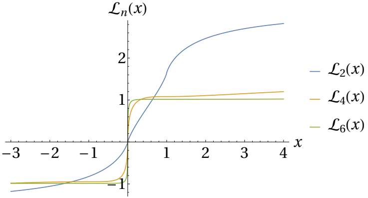

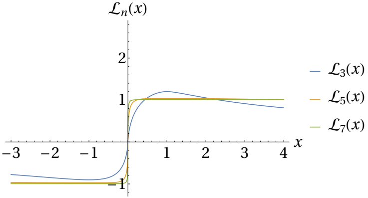

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Expansion of $\mathcal{F}_\pm(\varepsilon;x)$ on entire real axis

$$\mathcal{F}_\pm(\varepsilon;x) = \mathrm{e}^{\mathrm{i}\pi\varepsilon\,\mathrm{sgn}_{123}\,\Theta(-x)}\mathfrak{F}(\varepsilon;x) \mp \mathrm{i}\pi\varepsilon\,\Theta(x-1)\,,$$

where $\mathfrak{F}(\varepsilon;x) \equiv 1 + \ln\left|\frac{x}{x-1}\right| \sum_{n=1}^{\infty} \frac{(-\varepsilon)^n}{n!} \ln^{n-1}|x| - \sum_{n=2}^{\infty} \varepsilon^n \mathcal{L}_n(x)$

- imaginary part explicit
- all functions of real variable x manifestly real
- $\mathfrak{F}$ finite for $x \to \pm\infty$ in every order in ε

Collect $\ln|x|$ terms to obtain

$$\mathfrak{F}(\varepsilon;x) = |x|^{-\varepsilon} + \varepsilon \ln|1-x| - \sum_{n=2}^{\infty} \varepsilon^n \left[\frac{(-1)^n \ln|1-x| \ln^{n-1}|x|}{n!} + \mathcal{L}_n(x)\right]$$

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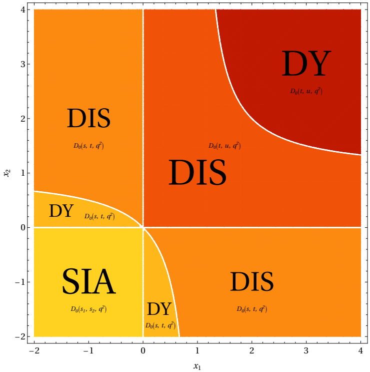

- logarithmic divergence for $x_1, x_2 = 1$ in DIS region cancels between 3 $\mathcal{F}$-functions
- spurious branch cuts $\mp i\pi\varepsilon\Theta(x-1)$ cancel in all kinematic regions

$$\begin{aligned}\mathcal{D}_0(\varepsilon; x_1, x_2) &= e^{i\pi\varepsilon\,\mathrm{sgn}_{123}\Theta(-x_1)}\mathfrak{F}(\varepsilon; x_1) + e^{i\pi\varepsilon\,\mathrm{sgn}_{123}\Theta(-x_2)}\mathfrak{F}(\varepsilon; x_2)\\ &\quad - e^{i\pi\varepsilon\,\mathrm{sgn}_{123}\Theta(-x_1-x_2+x_1x_2)}\mathfrak{F}(\varepsilon; x_1+x_2-x_1x_2)\end{aligned}$$

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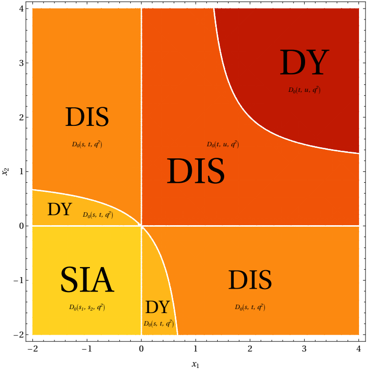

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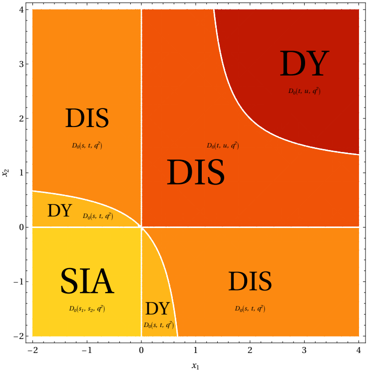
$$\mathcal{D}_0(\varepsilon; x_1, x_2) = e^{i\pi\varepsilon\,\mathrm{sgn}_{123}\Theta(-x_1)}\mathfrak{F}(\varepsilon; x_1) + e^{i\pi\varepsilon\,\mathrm{sgn}_{123}\Theta(-x_2)}\mathfrak{F}(\varepsilon; x_2) - e^{i\pi\varepsilon\,\mathrm{sgn}_{123}\Theta(-x_1-x_2+x_1x_2)}\mathfrak{F}(\varepsilon; x_1 + x_2 - x_1x_2)$$

Epsilon expansion of $\mathcal{D}_0(\varepsilon; x_1, x_2)$

$$\mathcal{D}_0(\varepsilon; x_1, x_2) = \mathcal{F}_\pm(\varepsilon; x_1) + \mathcal{F}_\mp(\varepsilon; x_2) - \mathcal{F}_+(\varepsilon; 1 - (1 - x_1)(1 - x_2))$$



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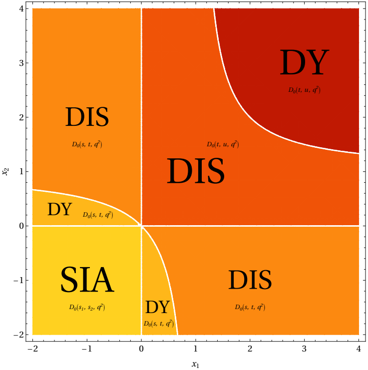

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Epsilon expansion of the scalar box integral

$$D_0\left(s_1, s_2, q^2\right) = \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2} \left|\frac{s_3 \mu^2}{s_1 s_2}\right|^{\varepsilon}$$
$$\times \Big[\left(\Theta(-s_2) + \Theta(s_2) e^{i\pi\varepsilon}\right) \mathfrak{F}(\varepsilon; x_1) + \left(\Theta(-s_1) + \Theta(s_1) e^{i\pi\varepsilon}\right) \mathfrak{F}(\varepsilon; x_2)$$
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Alternative representation of ε-expansion

Bring factor $\left|s_3\mu^2/s_1 s_2\right|^\varepsilon$ into square brackets

$$D_0\left(s_1, s_2, q^2\right) = \frac{1}{\varepsilon^2}\frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}\frac{2}{s_1 s_2}$$
$$\times\left[\left(\frac{\mu^2}{-s_2-\mathrm{i}0}\right)^\varepsilon \left|\frac{s_3}{s_1}\right|^\varepsilon \mathfrak{F}\left(\varepsilon; -\frac{s_3}{s_1}\right) + \left(\frac{\mu^2}{-s_1-\mathrm{i}0}\right)^\varepsilon \left|\frac{s_3}{s_2}\right|^\varepsilon \mathfrak{F}\left(\varepsilon; -\frac{s_3}{s_2}\right)\right.$$
$$\left. - \left(\frac{\mu^2}{-q^2-\mathrm{i}0}\right)^\varepsilon \left|\frac{s_3 q^2}{s_1 s_2}\right|^\varepsilon \mathfrak{F}\left(\varepsilon; -\frac{s_3 q^2}{s_1 s_2}\right)\right]$$

- Compared to result in terms of hypergeometric functions, ${}_2F_1$ was replaced by a single-valued version,

$${}_2F_1\left(1, -\varepsilon, 1-\varepsilon; x \pm \mathrm{i}\tilde{0}\right) \to |x|^\varepsilon\, \mathfrak{F}(\varepsilon; x)$$

Result in terms of hypergeometric functions

$$D_0\left(s_1, s_2, q^2\right) = \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2}$$
$$\times \left\{ \left[\frac{\mu^2}{-s_2 - \mathrm{i}0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon;\, -\frac{s_3}{s_1} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_2} - \frac{s_3}{s_1}\right)\right)\right.$$
$$+ \left[\frac{\mu^2}{-s_1 - \mathrm{i}0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon;\, -\frac{s_3}{s_2} + \mathrm{i}\tilde{0}\,\mathrm{sgn}\left(\frac{s_3}{s_1} - \frac{s_3}{s_2}\right)\right)$$
$$\left. - \left[\frac{\mu^2}{-q^2 - \mathrm{i}0}\right]^{\varepsilon} {}_2F_1\left(1, -\varepsilon, 1-\varepsilon;\, -\frac{s_3 q^2}{s_1 s_2} + \mathrm{i}\tilde{0}\right)\right\}$$

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$${}_2\mathrm{F}_1\left(1,-\varepsilon,1-\varepsilon;x\pm\mathrm{i}\tilde{0}\right)\to|x|^\varepsilon\,\mathfrak{F}(\varepsilon;x)$$

Epsilon expansion in the non-adjacent double off-shell case

$$
\begin{aligned}
D_0\left(s_1, s_2, p_2^2, p_4^2\right) = \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)\,\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \frac{2}{s_1 s_2 - p_2^2 p_4^2} \left| \frac{(p_2^2 + p_4^2 - s_1 - s_2)\mu^2}{s_1 s_2 - p_2^2 p_4^2} \right|^{\varepsilon} \\
\times \Bigg\{ \left(\Theta(-s_1) + \Theta(s_1) e^{i\pi\varepsilon}\right) \mathfrak{F}\left(\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_1}{s_1 s_2 - p_2^2 p_4^2}\right) \\
+ \left(\Theta(-s_2) + \Theta(s_2) e^{i\pi\varepsilon}\right) \mathfrak{F}\left(\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)s_2}{s_1 s_2 - p_2^2 p_4^2}\right) \\
- \left(\Theta(-p_2^2) + \Theta(p_2^2) e^{i\pi\varepsilon}\right) \mathfrak{F}\left(\varepsilon; -\frac{(p_2^2 + p_4^2 - s_1 - s_2)p_2^2}{s_1 s_2 - p_2^2 p_4^2}\right) \\
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\end{aligned}
$$

- All-order ε-expansion (not previously known)
- Real and imaginary parts can be easily read off
- Valid in all kinematic regions ($s_1, s_2, p_2^2, p_4^2 \in \mathbb{R}$)
- No spurious branch cuts

5. Conclusion and Outlook

Conclusion and Outlook

- All-order ε-expansion in terms of new single-valued polylogarithms
- All-order ε-expansion of non-adjacent double-off shell box not previously known
- Real and imaginary parts to all orders in ε
- Results free of spurious branch cuts
- Any kinematic divergences contained in logarithms, as single-valued polylogarithms are bounded

- Additional methods needed for further generalization of results
- Differential equations [Gehrmann and Remiddi, 2000], Mellin-Barnes integrals [Smirnov, 1999], negative dimensions [Anastasiou et al., 2000], recurrence relations w.r.t. d [Fleischer et al., 2003]
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