TMD factorization beyond the leading power

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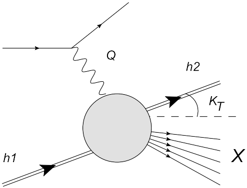
Meeting of the Forschergruppe FOR2926

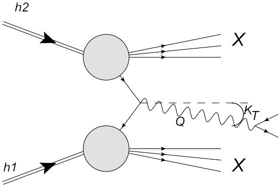
Transverse momentum dependent factorization

$$\frac{d\sigma}{dq_T} \quad \simeq \quad \sigma_0 \int \frac{d^2b}{(2\pi)^2} e^{-i(bq_T)} |C_V(Q)|^2 F_1(x_1,b;Q,Q^2) F_2(x_2,b;Q,Q^2)$$

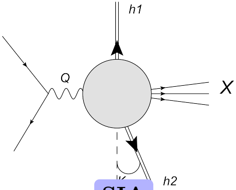
LP term is studied VERY WELL!



SIDIS



Drell-Yan

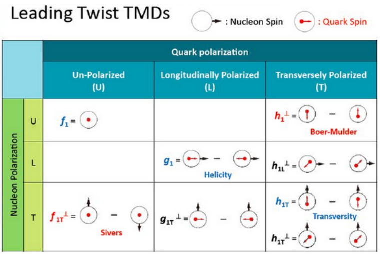


SIA

$q^2 = \pm Q^2$ momentum of hard probe

q_T^μ transverse component

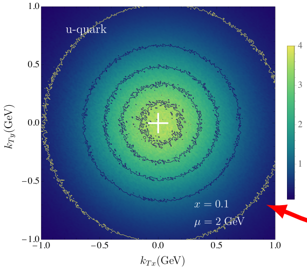
$$\begin{cases} Q^2 \gg \text{anything} \\ q_T^2 \text{ fixed} \end{cases}$$



Leading Twist TMDs

: Nucleon Spin : Quark Spin

Nucleon Polarization		Quark polarization		
		Un-Polarized (U)	Longitudinally Polarized (L)	Transversely Polarized (T)
	U	$f_1 =$		$h_1^\perp =$ − Boer-Mulder
	L		$g_1 =$ − Helicity	$h_{1L}^\perp =$ −
	T	$f_{1T}^\perp =$ − Sivers	$g_{1T}^\perp =$ −	$h_{1T} =$ − Transversity $h_{1T}^\perp =$ −



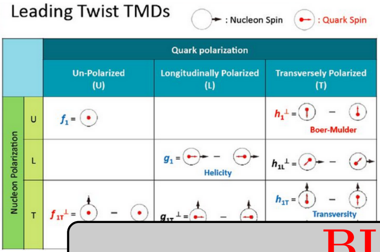
With N^{3}LO evolution

LP is very well studied

- Factorization theorem proven
- Multiple phenomenological tests
- Perturbative precision
    - Hard function: N^{4}LO
    - Evolution: N^{3}LO
    - OPE for components
        - CS kernel N^{4}LO
        - f_1 TMPDPDF N^{3}LO
        - d_1 TMDFF N^{3}LO
        - …

N^{4}LL for unpolarized

see talk by Valentin Moos tommorow



Leading Twist TMDs

: Nucleon Spin : Quark Spin

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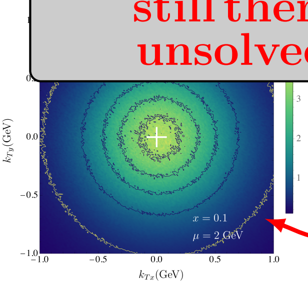
LP is very well studied

- Factorization theorem proven
- Multiple phenomenological tests
- Perturbative precision
  - Hard function: N^{4}LO
  - ^{3}LO
  - ponents
  - l N^{4}LO
  - DPDF N^{3}LO
  - FF N^{3}LO

N^{4}LL for unpolarized

see talk by Valentin Moos tommorow

BUT
still there are many unsolved problems



With N^{3}LO evolution

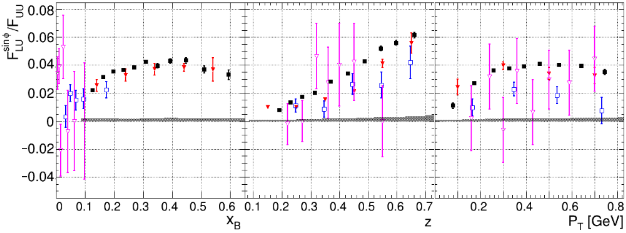
► Sub-leading power observables

$$F_{LU}^{\sin\phi_h} = \frac{2M}{Q}\mathcal{C}\left[-\frac{\hat{h}\cdot k_T}{M_h}\left(x\overset{\text{twist-3 pdf}}{\boxed{e}}\underset{\text{Collins FF}}{\boxed{H_1^\perp}}+\frac{M_h}{M}\overset{\text{unpolarized PDF}}{\boxed{f_1}}\frac{\boxed{\tilde{G}^\perp}}{\underset{\text{twist-3 FF}}{z}}\right)+\frac{\hat{h}\cdot p_T}{M}\left(x\overset{\text{twist-3 t-odd PDF}}{\boxed{g^\perp}}\underset{\text{unpolarized FF}}{\boxed{D_1}}+\frac{M_h}{M}\overset{\text{Boer-Mulders}}{\boxed{h_1^\perp}}\frac{\boxed{\tilde{E}}}{\underset{\text{twist-3 FF}}{z}}\right)\right]$$

by Timothy B. Hayward at QCD-N

SIDIS polarized structure
To describe it, one needs TMD
factorization at NLP.

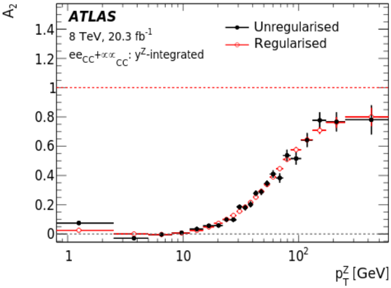
[CLAS, 2101.03544]

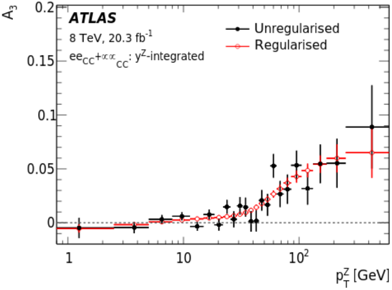


- **Sub-leading power observables**

$$\frac{d\sigma}{dp_T^Z\, dy^Z\, dm^Z\, d\cos\theta\, d\phi} = \frac{3}{16\pi}\frac{d\sigma^{U+L}}{dp_T^Z\, dy^Z\, dm^Z}\Big\{(1+\cos^2\theta)+\frac{1}{2}A_0(1-3\cos^2\theta)+A_1\ \sin 2\theta\ \cos\phi$$
$$+\frac{1}{2}A_2\ \sin^2\theta\ \cos 2\phi + A_3\ \sin\theta\ \cos\phi + A_4\ \cos\theta$$
$$+A_5\ \sin^2\theta\ \sin 2\phi + A_6\ \sin 2\theta\ \sin\phi + A_7\ \sin\theta\ \sin\phi\Big\}.$$

DY angular coefficients
To describe it, one needs TMD factorization at NNLP.

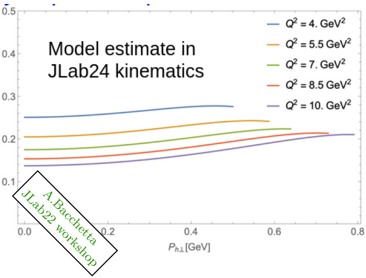




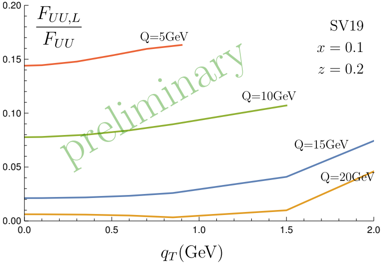
- **Sub-leading power observables**

$$\frac{d\sigma}{dx\,dy\,d\psi\,dz\,d\phi_h\,dP_{h\perp}^2} = \frac{\alpha^2}{xyQ^2}\,\frac{y^2}{2\,(1-\varepsilon)}\left(1+\frac{\gamma^2}{2x}\right)\left\{F_{UU,T}+\varepsilon F_{UU,L}+\sqrt{2\,\varepsilon(1+\varepsilon)}\,\cos\phi_h\,F_{UU}^{\cos\phi_h}\right.$$

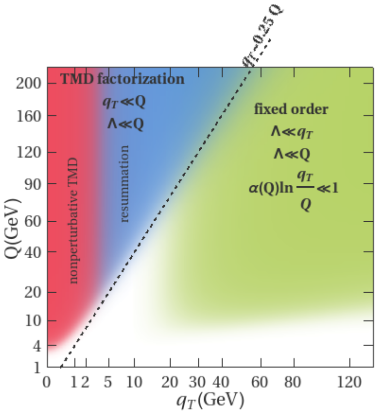
Longitudinal photons
To describe it, one needs TMD factorization at NNLP.



A.Bacchetta JLab22 workshop



- **Increase of applicability domain**



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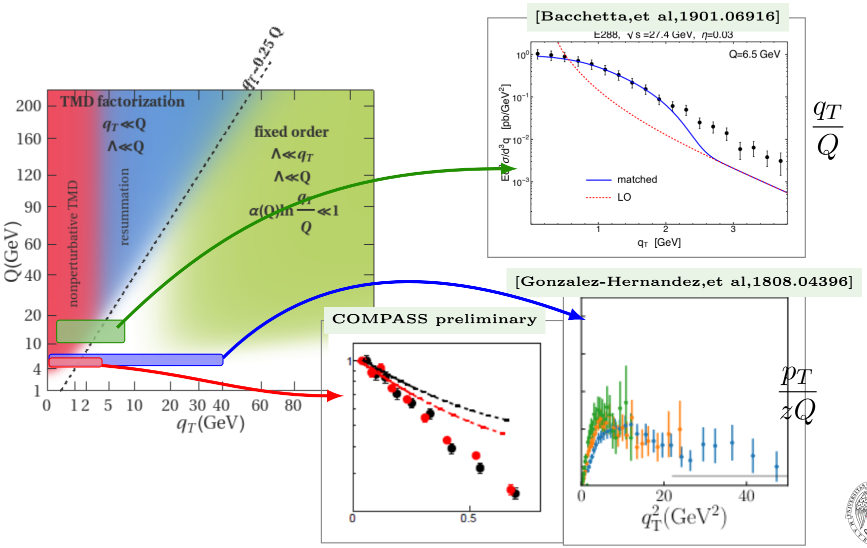


[Bacchetta,et al,1901.06916]

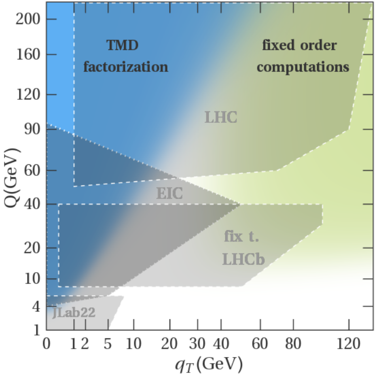
$\frac{q_T}{Q}$

[Gonzalez-Hernandez,et al,1808.04396]

COMPASS preliminary

$\frac{p_T}{zQ}$

- **Increase of applicability domain**



Phase space of EIC is centered directly in the transition region

COMPASS, JLab
fixed target LHCb
have large contribution of power corrections

Answer to all these problems

POWER CORRECTIONS

$$
\begin{aligned}
\frac{d\sigma}{dq_T} \simeq\ & \sigma_0 \int \frac{d^2b}{(2\pi)^2} e^{-i(bq_T)} \Big\{ |C_V(Q)|^2 F_1(x_1, b; Q, Q^2) F_2(x_2, b; Q, Q^2) & \longleftarrow \text{LP} \\
& + \left(\frac{q_T}{Q}; \frac{k_T}{Q}; \frac{M}{Q} \right) [C_2(Q) \otimes F_3(x, b; Q, Q^2) F_4(x, b; Q, Q^2)](x_1, x_2) & \longleftarrow \text{NLP} \\
& + \left(\frac{q_T^2}{Q^2}; \frac{k_T q_T}{Q^2}; \ldots \right) [C_3(Q) \otimes F_5(x, b; Q, Q^2) F_6(x, b; Q, Q^2)](x_1, x_2) & \longleftarrow \text{NNLP} \\
& + \ldots
\end{aligned}
$$

Outline

- Types of power corrections
- Factorization at NLP
- Beyond NLP

Four types of power corrections in TMD factorization

① $\frac{k_T}{Q}$ corrections ⇔ kinematic power corrections (KPC)

Derivatives of TMDs with respect to b^μ ⇔ tot.derivatives of semi-compact operators

$$\frac{i}{Q}\int d^2 b e^{-i(qb)_T} F_1(x_1, b; Q, Q^2)\frac{\partial}{\partial b_\mu} F_2(x_2, b; Q, Q^2)$$

$$\int d^2\mathbf{k}_1 d^2\mathbf{k}_2 \delta^{(2)}(\mathbf{q}_T - \mathbf{k}_1 - \mathbf{k}_2)\frac{\mathbf{k}_2^\mu}{Q} F_1(x_1, \mathbf{k}_1; Q, Q^2) F_2(x_2, \mathbf{k}_2; Q, Q^2)$$

Four types of power corrections in TMD factorization

② $\frac{\Lambda_{\text{QCD}}}{Q}$ corrections $\Leftrightarrow$ genuine power corrections

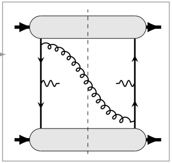
Higher twist TMD distributions

$$\int d^2b e^{-i(qb)_T} \int_{-1+x_2}^{1} \frac{dy}{y} F_1(x_1, b; Q, Q^2) \Phi_{\oplus}(-x-y, y, x, b; Q, Q^2)$$

Four types of power corrections in TMD factorization

③ $\frac{q_T}{Q}$ corrections $\Leftrightarrow$ $\frac{1}{bQ}$ corrections

Several sources



$$\boxed{\lim_{b\to 0} F_{\text{tw-N}}(x,b) = \frac{\alpha_s}{b^{N-1}} F_{\text{tw-2}}(x,b)}$$

$$\int d^2 b e^{-i(qb)_T} \frac{1}{b^2 Q^2} F_1(x_1, b; Q, Q^2) F_2(x_2, b; Q, Q^2)$$

$$\int d^2\mathbf{k}_1 d^2\mathbf{k}_2 \delta^{(2)}(\mathbf{q}_T - \mathbf{k}_1 - \mathbf{k}_2) \frac{\mathbf{q}_T^2}{Q^2} F_1(x_1, \mathbf{k}_1; Q, Q^2) F_2(x_2, \mathbf{k}_2; Q, Q^2)$$

Four types of power corrections in TMD factorization

④ Target mass corrections

Simplest?
Not systematically studied

Ignore it today

$$
\begin{aligned}
W^{\mu\nu} = \frac{1}{N_c}\int \frac{d^2b}{(2\pi)^2}\, e^{-i(q_T b)} \Bigg\{ & \\
& \Phi_2 \times \Phi_2 \\
+ \frac{1}{Q}\Big(& D\ \Phi_2 \times \Phi_2 + \quad \Phi_2 \times \Phi_3\Big) \\
+ \frac{1}{Q^2}\Big(& D^2\Phi_2 \times \Phi_2 + D\ \Phi_2 \times \Phi_3 + \quad \Phi_3 \times \Phi_3 + \quad \Phi_2 \times \Phi_4 + \frac{\Phi_2 \times \Phi_2}{b^2}\Big) \\
+ \frac{1}{Q^3}\Big(& D^3\Phi_2 \times \Phi_2 + \quad ... \quad + \quad \Phi_3 \times \Phi_4 + \quad ... \quad + \frac{D\Phi_2 \times \Phi_2}{b^2}\Big) \\
+ \frac{1}{Q^4}\Big(& D^4\Phi_2 \times \Phi_2 + \quad ... \quad + \quad \Phi_2 \times \Phi_5 + \quad ... \quad + \frac{D\Phi_2 \times \Phi_2}{b^2} + \frac{\Phi_2 \times \Phi_2}{b^4}\Big) \\
+ \cdots \Bigg\}^{\mu\nu}, &
\end{aligned}
$$

From the (twist-2) *resummation* perspective all types of corrections are alike $\sim (bQ)^{-n}$

Kinematic power corrections

$$\partial_\mu f_1(x,b) \longrightarrow \partial_\mu \left(1 + a_s\left(-2p(x)\ln(b^2\mu^2) + ...\right) + ...\right) \otimes q(x) = -4b^\mu \frac{a_s}{b^2}[p\otimes q](x) + ... \tag{1}$$

Genuine power corrections

- TMD twist-3 matches collinear twist-2 [S.Rodini, AV, 2204.03856]

$$f_\ominus^\perp(x_1,x_2,x_3,b) = \frac{2a_s}{M^2b^2}\Big[-C_F\frac{x_1-x_3}{x_1}f_1(-x_1)\left(\theta(x_2,x_3)-\theta(-x_2,-x_3)\right) + T_F\frac{x_1-x_3}{x_2}f_g(-x_2)\left(\theta(x_1,x_3)-\theta(-x_1,-x_3)\right)\Big] + \mathcal{O}\left(b^2,\frac{a_s^2}{b^2}\right),$$

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Kinematic power corrections

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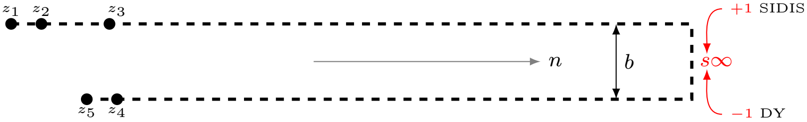
$$+T_F\frac{x_1-x_3}{x_2}f_g(-x_2)\left(\theta(x_1,x_3)-\theta(-x_1,-x_3)\right)\Big] + \mathcal{O}\left(b^2, \frac{a_s^2}{b^2}\right),$$

BUT they do not mix in the properly organized TMD factorization.
$\Longrightarrow$ TMD-twist

TMD-twist

Any TMD operator is the product of two *semi-compact* operators

$$\mathcal{O}_{NM}(\{z_1,...,z_n\},b)=U_N(\{z_1,...\},b)U_M(\{...,z_n\},b)$$



$$\mu^2\frac{d}{d\mu^2}\mathcal{O}_{NM}(\{z_1,...,z_n\},b;\mu,\zeta)=(\gamma_N(\{z_1,...\})+\gamma_M(\{...,z_n\}))\otimes\mathcal{O}_{NM}(\{z_1,...,z_n\},b;\mu,\zeta)$$

$$\zeta\frac{d}{d\zeta}\mathcal{O}_{NM}(\{z_1,...,z_n\},b;\mu,\zeta)=-\mathcal{D}_{NM}\mathcal{O}_{NM}(\{z_1,...,z_n\},b;\mu,\zeta)+??$$

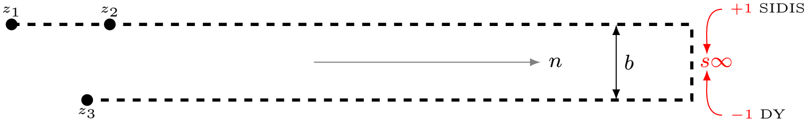
TMD operators with different TMD-twist
do not mix under evolution
(at least UV evolution)

TMD-twist-(1,1)

Usual TMDs

$U_1 = [..]\xi =$ good-component of quark field (twist-1)

$$\widetilde{\Phi}^{[\Gamma]}_{11}(\{z_1,z_2\},b)=\langle p,s|\bar{\xi}(z_1 n+b)..\frac{\Gamma}{2}..\xi(z_2 n)|p,s\rangle$$



$$\begin{aligned}
\mu^2\frac{d}{d\mu^2}\widetilde{\Phi}_{11}(\{z_{1,2}\},b;\mu,\zeta) &= \left(\widetilde{\gamma}_1(z_1,\mu,\zeta)+\widetilde{\gamma}_1(z_2,\mu,\zeta)\right)\widetilde{\Phi}_{11}(\{z_{1,2}\},b;\mu,\zeta)\\
\zeta\frac{d}{d\zeta}\widetilde{\Phi}_{11}(\{z_{1,2}\},b;\mu,\zeta) &= -\mathcal{D}(b,\mu)\widetilde{\Phi}_{11}(\{z_{1,2}\},b;\mu,\zeta)
\end{aligned}$$

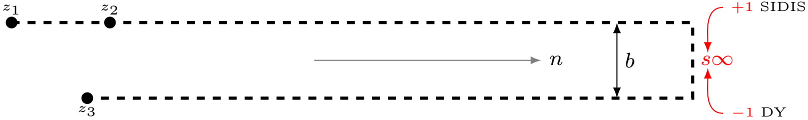
- $\gamma_1 =$ anomalous dimension of U_1 ($\mathrm{N}^3\mathrm{LO}$)
- $\mathcal{D} =$ CS kernel (NP)

TMD-twist-(2,1)

Appear at NLP

$U_1 = [..]\xi =$ good-component of quark field (twist-1)
$U_2 = [..]F_{\mu+}[..]\xi =$ good-components of gluon and quark fields (twist-2)

$$\widetilde{\Phi}_{21}^{[\Gamma]}(\{z_1,z_2,z_3\},b) = \langle p,s|\bar{\xi}(z_1 n+b)..F_{\mu+}(z_2 n+b)..\frac{\Gamma}{2}..\xi(z_3 n)|p,s\rangle$$



$$\mu^2\frac{d}{d\mu^2}\widetilde{\Phi}_{21}(\{z_{1,2,3}\},b;\mu,\zeta) = (\widetilde{\gamma}_2(z_1,z_2,\mu,\zeta)+\widetilde{\gamma}_1(z_3,\mu,\zeta))\,\widetilde{\Phi}_{21}(\{z_{1,2,3}\},b;\mu,\zeta)$$

$$\zeta\frac{d}{d\zeta}\widetilde{\Phi}_{21}(\{z_{1,2,3}\},b;\mu,\zeta) = -\mathcal{D}(b,\mu)\widetilde{\Phi}_{21}(\{z_{1,2,3}\},b;\mu,\zeta)$$

- $\gamma_1 =$ anomalous dimension of U_1 (N^{3}LO)
- $\gamma_2 =$ anomalous dimension of U_2 (LO)
- $\mathcal{D} =$ CS kernel (NP)

Similar for TMD-twist-(1,2)

Making story short: we introduce real/T-definite combination of operator and parametrize them

- **32 distributions** ($\bullet = \oplus$ and $\ominus$)
- **16 T-odd** and **16 T-even**

Example

$$\begin{aligned}\Phi_{\bullet}^{\mu[\gamma^+]}(x_{1,2,3},b) &= \epsilon^{\mu\nu}s_{T\nu}Mf_{\bullet T}(x_{1,2,3},b)+ib^{\mu}M^2f_{\bullet}^{\perp}(x_{1,2,3},b)\\ &+i\lambda\epsilon^{\mu\nu}b_{\nu}M^2f_{\bullet L}^{\perp}(x_{1,2,3},b)+b^2M^3\epsilon_T^{\mu\nu}\left(\frac{g_{T,\nu\rho}}{2}-\frac{b_{\nu}b_{\rho}}{b^2}\right)s_T^{\rho}f_{\bullet T}^{\perp}(x_{1,2,3},b)\end{aligned}$$

$$f_{\oplus,T;DY}=f_{\oplus,T;SIDIS},\qquad f_{\oplus;DY}^{\perp}=-f_{\oplus;SIDIS}^{\perp}$$

	U	L	$\mathrm{T}_{J=0}$	$\mathrm{T}_{J=1}$	$\mathrm{T}_{J=2}$
U	$f_{\bullet}^{\perp}$	$g_{\bullet}^{\perp}$		$h_{\bullet}$	$h_{\bullet}^{\perp}$
L	$f_{\bullet L}^{\perp}$	$g_{\bullet L}^{\perp}$	$h_{\bullet L}$		$h_{\bullet L}^{\perp}$
T	$f_{\bullet T}$, $f_{\bullet T}^{\perp}$	$g_{\bullet T}$, $g_{\bullet T}^{\perp}$	$h_{\bullet T}^{D\perp}$	$h_{\bullet T}^{A\perp}$	$h_{\bullet T}^{S\perp}$, $h_{\bullet T}^{T\perp}$

Evolution equations mix T-even and T-odd distributions

[S.Rodini,AV,2204.03856]

$$\mu^2\frac{d}{d\mu^2}\left[\begin{array}{c}F_\oplus\\F_\ominus\end{array}\right] = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+\Upsilon_{x_1x_2x_3}\right)\left[\begin{array}{c}F_\oplus\\F_\ominus\end{array}\right] + \left[\begin{array}{cc}2\mathbb{P}_{x_2x_1} & 2\pi s\Theta_{x_1x_2x_3}\\-2\pi s\Theta_{x_1x_2x_3} & 2\mathbb{P}_{x_2x_1}\end{array}\right]\left[\begin{array}{c}F_\oplus\\F_\ominus\end{array}\right],$$

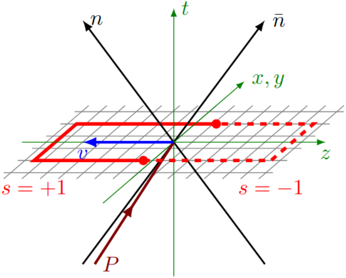
- Real functions = real evolution
- Mixing is proportional to s, so T-parity is preserved, and distributions are universal
- Similar structure in the coefficient function, such that all poles cancel
- All kernels are known at 1-loop

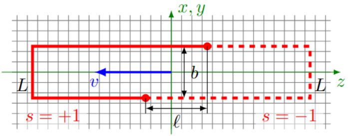
Skipping material

- TMD factorization at NLP

[AV, V.Moos, I.Scimemi,2109.09771]
[M.Ebert,A.Gao,I.Stewart, 2112.07680]
[S.Rodini, AV, 2211.04494]
[S.Rodini, AV, *in prep.*]

  - Derived and checked at NLO
  - Mechanism of special rapidity divergences and restoration of boost-invariance
  - Not the same as “naive” TMD factorization at NLP
  - Expression for cross-section is rather complicated (to be released soon)
  - Expression for quasi-TMDPDF is known





$$\Omega^{[\Gamma]}(x, b; \mu) = \int \frac{d\ell}{2\pi} e^{-ix\ell p^+} \langle p, s|\bar{q}(\ell n + b)[\text{staple along v}]\Gamma q(0)|p, s\rangle$$

- $\Gamma = \{\gamma^+, \gamma^+\gamma^5, \sigma^{\alpha+}\gamma^5\}$ LP factorization [Ji,Luo,...,Stewart, Ebert,...,19-20]
- $\Gamma = \{1, \gamma^5, \gamma_T^\alpha, \sigma^{+-}, ...\}$ NLP factorization [S.Rodini, AV, 2211.04494]
- $\Gamma = \{\gamma^-, \gamma^-\gamma^5, \sigma^{\alpha-}\gamma^5\}$ NNLP factorization factorization

$$
\begin{aligned}
\Omega^{[\Gamma]}(x,b,\mu) = {} & \Psi(b;\mu,\bar{\zeta})\mathbb{C}_{11}\Phi_{11}^{[\![\Gamma]\!]}(x,b;\mu,\zeta) \\
& + \frac{i}{2xP_+}\mathbb{C}_{11}\Psi(b)\left(\partial_\mu - \frac{1}{2}[\partial_\mu \mathcal{D}(b,\mu)]\ln\left(\frac{\bar{\zeta}}{\zeta}\right)\right)\Phi_{11}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(x,b;\mu,\zeta) \\
& + \frac{1}{2xP_+}\mathbb{C}_{11v}\boldsymbol{\Psi}^{(0)}_{\mu,21}(b;\mu,\bar{\zeta})\Phi_{11}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(x,b;\mu,\zeta) \\
& + \frac{i}{2xP_+}\Psi(b;\mu,\bar{\zeta})\int_{-1}^{1}dx_2\Big[\mathbb{C}_R(x,x_2)\boldsymbol{\Phi}_{\mu,\oplus}^{[\![\gamma^\mu\gamma^+\Gamma-\Gamma\gamma^+\gamma^\mu]\!]}(\tilde{x},b;\mu,\zeta) \\
& \qquad + s\pi\mathbb{C}_I(x,x_2)\boldsymbol{\Phi}_{\mu,\ominus}^{[\![\gamma^\mu\gamma^+\Gamma-\Gamma\gamma^+\gamma^\mu]\!]}(\tilde{x},b;\mu,\zeta) \\
& \qquad - i\mathbb{C}_R(x,x_2)\boldsymbol{\Phi}_{\mu,\ominus}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(\tilde{x},b;\mu,\zeta) \\
& \qquad + is\pi\mathbb{C}_I(x,x_2)\boldsymbol{\Phi}_{\mu,\oplus}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(\tilde{x},b;\mu,\zeta)\Big] \\
& + \mathcal{O}(\lambda^2),
\end{aligned}
\tag{4.48}
$$

$$\Omega^{[\Gamma]}(x,b,\mu) = \boxed{\Psi(b;\mu,\bar{\zeta})\mathbb{C}_{11}\Phi_{11}^{[\![\Gamma]\!]}(x,b;\mu,\zeta)} \text{ — } \boxed{\textbf{LP term}} \qquad (4.48)$$

$$+\frac{i}{2xP_+}\mathbb{C}_{11}\Psi(b)\Big(\quad \cdots \quad {}^{\mu]\!]}(x,b;\mu,\zeta)$$

$$+\frac{1}{2xP_+}\mathbb{C}_{11v}\mathbf{\Psi}^{(0)}_{\mu,21} \quad \cdots$$

$$+\frac{i}{2xP_+}\Psi(b;\mu,\bar{\zeta})\int_{-1}^{1}dx_2\Big[\mathbb{C}_R(x,x_2)\mathbf{\Phi}_{\mu,\oplus}^{[\![\gamma^\mu\gamma^+\Gamma-\Gamma\gamma^+\gamma^\mu]\!]}(\tilde{x},b;\mu,\zeta)$$
$$+s\pi\mathbb{C}_I(x,x_2)\mathbf{\Phi}_{\mu,\ominus}^{[\![\gamma^\mu\gamma^+\Gamma-\Gamma\gamma^+\gamma^\mu]\!]}(\tilde{x},b;\mu,\zeta)$$
$$-i\mathbb{C}_R(x,x_2)\mathbf{\Phi}_{\mu,\ominus}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(\tilde{x},b;\mu,\zeta)$$
$$+is\pi\mathbb{C}_I(x,x_2)\mathbf{\Phi}_{\mu,\oplus}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(\tilde{x},b;\mu,\zeta)\Big]$$

$$+\mathcal{O}(\lambda^2),$$

- ▶ Φ_{11} is physical TMDPDF
- ▶ Ψ is “lattice”-specific function

$$
\begin{aligned}
\Omega^{[\Gamma]}(x,b,\mu) = {} & \Psi(b;\mu,\bar\zeta)\mathbb{C}_{11}\Phi_{11}^{[\![\Gamma]\!]}(x,b;\mu,\zeta) \\
& \boxed{+\frac{i}{2xP_+}\mathbb{C}_{11}\Psi(b)\left(\partial_\mu-\frac{1}{2}[\partial_\mu\mathcal{D}(b,\mu)]\ln\left(\frac{\bar\zeta}{\zeta}\right)\right)\Phi_{11}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(x,b;\mu,\zeta)} \\
& +\frac{1}{2xP_+}\mathbb{C}_{11v}\boldsymbol{\Psi}^{(0)}_{\mu,21}(b;\mu,\bar\zeta)\Phi_{11}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(x,b;\mu,\zeta) \\
& +\frac{i}{2}\Psi(b;\mu,\bar\zeta)\int^1 dx_2\Big[\mathbb{C}_R(x,x_2)\boldsymbol{\Phi}^{[\![\gamma^\mu\gamma^+\Gamma-\Gamma\gamma^+\gamma^\mu]\!]}(\tilde x,b;\mu,\zeta) \\
& \qquad ;\mu,\zeta) \\
& \qquad \mu,\zeta) \\
& \qquad +is\pi\mathbb{C}_I(x,x_2)\boldsymbol{\Psi}_{\mu,\oplus} \qquad (x,b;\mu,\zeta)\Big] \\
& +\mathcal{O}(\lambda^2),
\end{aligned}
\tag{4.48}
$$

Kinematic power correction

- The “long derivative” is the result of cancelation of “special” rapidity divergences.
- Required for restoration of the boost-invariance $\zeta \to \alpha\zeta$, $\bar\zeta \to \bar\zeta/\alpha$

$$
\begin{aligned}
\Omega^{[\Gamma]}(x,b,\mu) = {} & \Psi(b;\mu,\bar\zeta)\mathbb{C}_{11}\Phi_{11}^{[\![\Gamma]\!]}(x,b;\mu,\zeta) \qquad (4.48)\\
& +\frac{i}{2xP_+}\mathbb{C}_{11}\Psi(b)\left(\partial_\mu-\frac{1}{2}[\partial_\mu\mathcal{D}(b,\mu)]\ln\left(\frac{\bar\zeta}{\zeta}\right)\right)\Phi_{11}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(x,b;\mu,\zeta)\\
& \boxed{+\frac{1}{2xP_+}\mathbb{C}_{11v}\boldsymbol{\Psi}^{(0)}_{\mu,21}(b;\mu,\bar\zeta)\Phi_{11}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(x,b;\mu,\zeta)} \text{ — } \boxed{\textbf{genuine NLP term}}\\
& +\frac{i}{2xP_+}\Psi(b;\mu,\bar\zeta)\int_{-1}^{1}dx_2\Big[\mathbb{C}_R(x,x\\
& \qquad\qquad +s\pi\mathbb{C}_I(x\\
& \qquad\qquad -i\mathbb{C}_R(x,x_2)\boldsymbol{\Phi}_{\mu,\ominus}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(\tilde x,b;\mu,\zeta)\\
& \qquad\qquad +is\pi\mathbb{C}_I(x,x_2)\boldsymbol{\Phi}_{\mu,\oplus}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(\tilde x,b;\mu,\zeta)\Big]\\
& +\mathcal{O}(\lambda^2),
\end{aligned}
$$

- ▶ Ψ_{21} is “lattice”-specific function of TMD-twist-3

Ω

- $\Phi_\oplus$ and $\Phi_\ominus$ are TMDPDFs of twist-3
- Two types of terms, with different coefficient functions
- s is the direction of the Wilson line

$$(4.48)$$

$${}^{+}\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(x,b;\mu,\zeta)$$

genuine NLP term

$$\boxed{\begin{aligned}+\frac{i}{2xP_+}\Psi(b;\mu,\bar\zeta)\int_{-1}^{1}dx_2\Big[&\mathbb{C}_R(x,x_2)\mathbf{\Phi}_{\mu,\oplus}^{[\![\gamma^\mu\gamma^+\Gamma-\Gamma\gamma^+\gamma^\mu]\!]}(\tilde x,b;\mu,\zeta)\\&+s\pi\mathbb{C}_I(x,x_2)\mathbf{\Phi}_{\mu,\ominus}^{[\![\gamma^\mu\gamma^+\Gamma-\Gamma\gamma^+\gamma^\mu]\!]}(\tilde x,b;\mu,\zeta)\\&-i\mathbb{C}_R(x,x_2)\mathbf{\Phi}_{\mu,\ominus}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(\tilde x,b;\mu,\zeta)\\&+is\pi\mathbb{C}_I(x,x_2)\mathbf{\Phi}_{\mu,\oplus}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(\tilde x,b;\mu,\zeta)\Big]\end{aligned}}$$

$$+\mathcal{O}(\lambda^2),$$

Ω (4.48)

$^{+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(x,b;\mu,\zeta)$

- $\Phi_\oplus$ and $\Phi_\ominus$ are TMDPDFs of twist-3
- Two types of terms, with different coefficient functions
- s is the direction of the Wilson line
- $\mathbb{C}_I \sim \delta(x_2)$, Qiu-Sterman-like contribution in TMD factorization

genuine NLP term

$$\boxed{\begin{aligned}+\frac{i}{2xP_+}\Psi(b;\mu,\bar\zeta)\int_{-1}^{1}dx_2\Big[&\mathbb{C}_R(x,x_2)\mathbf{\Phi}_{\mu,\oplus}^{[\![\gamma^\mu\gamma^+\Gamma-\Gamma\gamma^+\gamma^\mu]\!]}(\tilde{x},b;\mu,\zeta)\\ &\boxed{+s\pi\mathbb{C}_I(x,x_2)\mathbf{\Phi}_{\mu,\ominus}^{[\![\gamma^\mu\gamma^+\Gamma-\Gamma\gamma^+\gamma^\mu]\!]}(\tilde{x},b;\mu,\zeta)}\\ &-i\mathbb{C}_R(x,x_2)\mathbf{\Phi}_{\mu,\ominus}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(\tilde{x},b;\mu,\zeta)\\ &\boxed{+is\pi\mathbb{C}_I(x,x_2)\mathbf{\Phi}_{\mu,\oplus}^{[\![\gamma^\mu\gamma^+\Gamma+\Gamma\gamma^+\gamma^\mu]\!]}(\tilde{x},b;\mu,\zeta)}\Big]\end{aligned}}$$

$$+\mathcal{O}(\lambda^2),$$

Generally NLP factorization is very cumbersome, but for some channels it simplifies, and can be used practically, see [Hai-Tao Shu, M.Schlemmer, et al, 2302.06502]
also talk by Hai-Tao Shu (today)

$$
\begin{aligned}
W^{\mu\nu} = \frac{1}{N_c} \int \frac{d^2 b}{(2\pi)^2}\, e^{-i(q_T b)} \Bigg\{ \\
& \boxed{\Phi_2 \times \Phi_2 \qquad\qquad\qquad\qquad \text{LP} \checkmark} \\
& \boxed{+\frac{1}{Q}\Big(D\ \ \Phi_2 \times \Phi_2 + \quad \Phi_2 \times \Phi_3\Big) \quad \text{NLP} \checkmark} \\
& +\frac{1}{Q^2}\Big(D^2 \Phi_2 \times \Phi_2 + D\ \ \Phi_2 \times \Phi_3 + \quad \Phi_3 \times \Phi_3 + \quad \Phi_2 \times \Phi_4 + \frac{\Phi_2 \times \Phi_2}{b^2}\Big) \\
& +\frac{1}{Q^3}\Big(D^3 \Phi_2 \times \Phi_2 + \quad \dots \quad + \quad \Phi_3 \times \Phi_4 + \quad \dots \quad + \frac{D\Phi_2 \times \Phi_2}{b^2}\Big) \\
& +\frac{1}{Q^4}\Big(D^4 \Phi_2 \times \Phi_2 + \quad \dots \quad + \quad \Phi_2 \times \Phi_5 + \quad \dots \quad + \frac{D\Phi_2 \times \Phi_2}{b^2} + \frac{\Phi_2 \times \Phi_2}{b^4}\Big) \\
& +\cdots \Bigg\}^{\mu\nu},
\end{aligned}
$$

$$
\begin{aligned}
W^{\mu\nu} = \frac{1}{N_c}\int \frac{d^2b}{(2\pi)^2}\, e^{-i(q_T b)} \Big\{ & \Phi_2 \times \Phi_2 \\
& + \frac{1}{Q}\Big(D\ \Phi_2 \times \Phi_2 + \quad \Phi_2 \times \Phi_3\Big) \\
& + \frac{1}{Q^2}\Big(D^2 \Phi_2 \times \Phi_2 + D\ \Phi_2 \times \Phi_3 + \quad \Phi_3 \times \Phi_3 + \quad \Phi_2 \times \Phi_4 + \frac{\Phi_2 \times \Phi_2}{b^2}\Big) \\
& + \frac{1}{Q^3}\Big(D^3 \Phi_2 \times \Phi_2 + \quad \ldots \quad + \quad \Phi_3 \times \Phi_4 + \quad \ldots \quad + \frac{D\Phi_2 \times \Phi_2}{b^2}\Big) \\
& + \frac{1}{Q^4}\Big(D^4 \Phi_2 \times \Phi_2 + \quad \ldots \quad + \quad \Phi_2 \times \Phi_5 + \quad \ldots \quad + \frac{D\Phi_2 \times \Phi_2}{b^2} + \frac{\Phi_2 \times \Phi_2}{b^4}\Big) \\
& + \cdots \Big\}^{\mu\nu},
\end{aligned}
$$

resummed KPC
in progress

- Resporation of gauge and Lorenz invariance
- Non-vanishing $q_T = 0$, and larger than $\frac{\Lambda}{Q}$
- Alike LP expression, but with a different convolution integral

$$
\begin{aligned}
W^{\mu\nu} = \frac{1}{N_c}\int \frac{d^2b}{(2\pi)^2}\, e^{-i(q_T b)} \Bigg\{ & \\
& \Phi_2 \times \Phi_2 \\
& + \frac{1}{Q}\Big(D\ \Phi_2 \times \Phi_2 + \Phi_2 \times \Phi_3\Big) + \frac{b}{b^2}\Phi_2 \times \Phi_2 \\
& + \frac{1}{Q^2}\Big(D^2\Phi_2 \times \Phi_2 + D\ \Phi_2 \times \Phi_3 + \Phi_3 \times \Phi_3 + \Phi_2 \times \Phi_4 + \frac{\Phi_2 \times \Phi_2}{b^2}\Big) \\
& + \frac{1}{Q^3}\Big(D^3\Phi_2 \times \Phi_2 + \ldots + \Phi_3 \times \Phi_4 + \ldots + \frac{D\Phi_2 \times \Phi_2}{b^2}\Big) + \frac{b}{b^4}\Phi_2 \times \Phi_2 \\
& + \frac{1}{Q^4}\Big(D^4\Phi_2 \times \Phi_2 + \ldots + \Phi_2 \times \Phi_5 + \ldots + \frac{D\Phi_2 \times \Phi_2}{b^2} + \frac{\Phi_2 \times \Phi_2}{b^4}\Big) \\
& + \cdots \Bigg\}^{\mu\nu},
\end{aligned}
$$

- Same NP content as at LP
- Looks conceptually possible

leading $\frac{q_T}{Q}$ possible?

Resummed KPC

$$\frac{d\sigma}{dq_T} = |C_V(Q)|^2 \int d^2\mathbf{k}_1 d^2\mathbf{k}_2 \delta(\mathbf{q}_T - \mathbf{k}_1 - \mathbf{k}_2)\theta((\mathbf{k}_1 - \mathbf{k}_2)^2 < \tau^2)$$

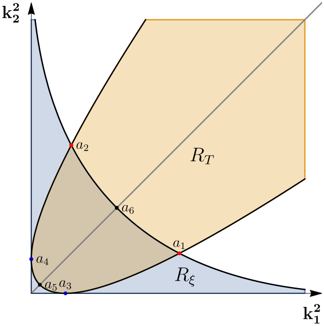
$$\frac{\rho_{pp}(\text{polarizations})}{\sqrt{\lambda(\mathbf{k}_1^2, \mathbf{k}_2^2, \tau^2)}} F(\xi_1(\mathbf{k}_{1,2}), \mathbf{k}_1^2, Q, Q^2) F(\xi_2(\mathbf{k}_{1,2}), \mathbf{k}_2^2, Q, Q^2)$$

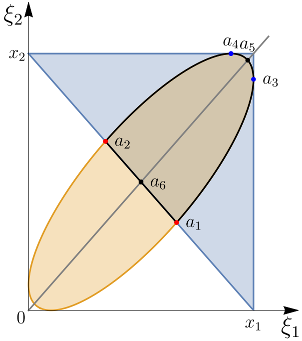
- Restore gauge and Lorenz invariance
- Full expression is raver cumbersome, but simplifies at $\zeta = \bar{\zeta} = Q^2$
- Can be resummed to a simpler expression at $\zeta = \bar{\zeta} = Q^2$ and in momentum space

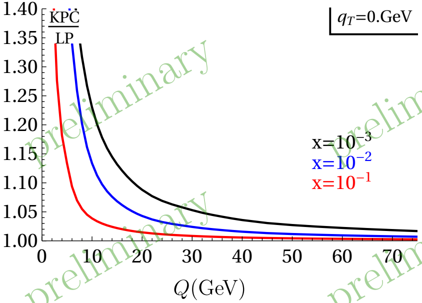
Resummed KPC

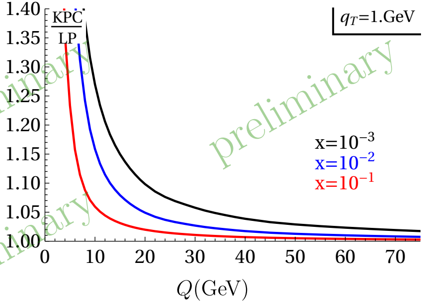
$$\frac{d\sigma}{dq_T} = |C_V(Q)|^2 \int d^2\mathbf{k}_1 d^2\mathbf{k}_2 \delta(\mathbf{q}_T - \mathbf{k}_1 - \mathbf{k}_2)\theta((\mathbf{k}_1 - \mathbf{k}_2)^2 < \tau^2)$$

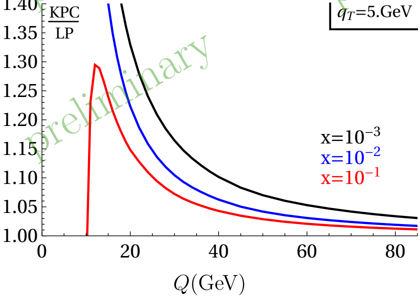
$$\frac{\rho_{pp}(\text{polarizations})}{\sqrt{\lambda(\mathbf{k}_1^2, \mathbf{k}_2^2, \tau^2)}} F(\xi_1(\mathbf{k}_{1,2}), \mathbf{k}_1^2, Q, Q^2) F(\xi_2(\mathbf{k}_{1,2}), \mathbf{k}_2^2, Q, Q^2)$$











- Mostly change the normalization
- $\sim 1\%$ at LHC energy
- $\sim 50 - 100\%$ at HERMES

Conclusion

TMD factorization at NLP

- Operator expression at NLP/NLO is known
- Full classification is done
- Restoration of EM-conservation
- Also for qTMDs

TMD factorization beyond NLP

- NNLP is done! (finalizing NLO)
- Singularities at $b \to 0$
- Applications?

Conclusion

TMD factorization at NLP

- Operator expression at NLP/NLO is known
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TMD factorization beyond NLP

- NNLP is done! (finalizing NLO)
- Singularities at $b \to 0$
- Applications?

Thank you for attention!

Backup

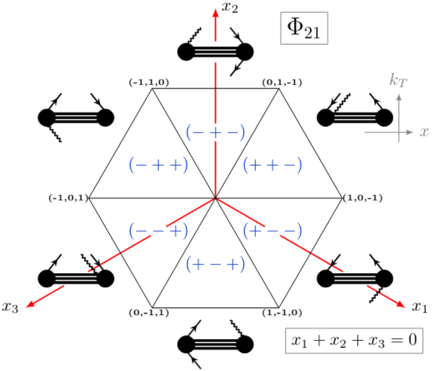
To momentum-fraction space

$$\widetilde{\Phi}^{[\Gamma]}_{11}(z_1,z_2,b) = p^+\int_{-1}^{1} dx e^{ix(z_1-z_2)p^+}\Phi^{[\Gamma]}_{11}(x,b),$$

$$\widetilde{\Phi}^{[\Gamma]}_{\mu,21}(z_1,z_2,z_3,b) = (p^+)^2\int [dx] e^{-i(x_1z_1+x_2z_2+x_3z_3)p^+}\Phi^{[\Gamma]}_{\mu,21}(x_1,x_2,x_3,b),$$

$$\widetilde{\Phi}^{[\Gamma]}_{\mu,12}(z_1,z_2,z_3,b) = (p^+)^2\int [dx] e^{-i(x_1z_1+x_2z_2+x_3z_3)p^+}\Phi^{[\Gamma]}_{\mu,12}(x_1,x_2,x_3,b),$$

$$\boxed{\int [dx] = \int_{-1}^{1} dx_1 \int_{-1}^{1} dx_2 \int_{-1}^{1} dx_3\, \delta(x_1+x_2+x_3)}$$



Support domain $|x_i| < 1$
momentum-fractions
could be **positive or negative**

some papers miss this point

- important for divergences-cancelation
- agreement with collinear evolution
- evolution mixture

Evolution equations for TMD-twist-(1,2) and (2,1) distribution

$$\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,21} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+\Upsilon_{x_1x_2x_3}+2\pi i\, s\,\Theta_{x_1x_2x_3}\right)\Phi^{[\Gamma]}_{\mu,21}$$
$$+\mathbb{P}^A_{x_2x_1}\otimes\Phi^{[\gamma^\nu\gamma^\mu\Gamma]}_{\nu,21}+\mathbb{P}^B_{x_2x_1}\otimes\Phi^{[\gamma^\mu\gamma^\nu\Gamma]}_{\nu,21},$$
$$\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,12} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right)+\Upsilon_{x_3x_2x_1}+2\pi i\, s\,\Theta_{x_3x_2x_1}\right)\Phi^{[\Gamma]}_{\mu,12}$$
$$+\mathbb{P}^A_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\mu\gamma^\nu]}_{\nu,12}+\mathbb{P}^B_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\nu\gamma^\mu]}_{\nu,12},$$

- $\mathbb{P}$ are BFLK kernels
- Υ contains $\ln(x)$
- Rapidity evolution is same as for TMD-twist-(1,1)

$$\zeta\frac{d}{d\zeta}\Phi^{[\Gamma]}_{\mu,12}(x_1,x_2,x_3,b;\mu,\zeta) = -\mathcal{D}(b,\mu)\Phi^{[\Gamma]}_{\mu,12}(x_1,x_2,x_3,b;\mu,\zeta),$$
$$\zeta\frac{d}{d\zeta}\Phi^{[\Gamma]}_{\mu,21}(x_1,x_2,x_3,b;\mu,\zeta) = -\mathcal{D}(b,\mu)\Phi^{[\Gamma]}_{\mu,21}(x_1,x_2,x_3,b;\mu,\zeta).$$

Evolution equations for TMD-twist-(1,2) and (2,1) distribution

$$\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,21} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_1x_2x_3} + \boxed{2\pi i\, s\,\Theta_{x_1x_2x_3}}\right)\Phi^{[\Gamma]}_{\mu,21}$$
$$+\mathbb{P}^A_{x_2x_1}\otimes\Phi^{[\gamma^\nu\gamma^\mu\Gamma]}_{\nu,21} + \mathbb{P}^B_{x_2x_1}\otimes\Phi^{[\gamma^\mu\gamma^\nu\Gamma]}_{\nu,21},$$

$$\mu^2\frac{d}{d\mu^2}\Phi^{[\Gamma]}_{\mu,12} = \left(\frac{\Gamma_{\text{cusp}}}{2}\ln\left(\frac{\mu^2}{\zeta}\right) + \Upsilon_{x_3x_2x_1} + \boxed{2\pi i\, s\,\Theta_{x_3x_2x_1}}\right)\Phi^{[\Gamma]}_{\mu,12}$$
$$+\mathbb{P}^A_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\mu\gamma^\nu]}_{\nu,12} + \mathbb{P}^B_{x_2x_3}\otimes\Phi^{[\Gamma\gamma^\nu\gamma^\mu]}_{\nu,12},$$

Imaginary-part of collinear logarithms

- Discontinious
- Process dependent!
- Complex evolution for complex functions

$$\Theta_{x_1x_2x_3} = a_s \times \begin{cases} \frac{C_A}{2} & x_{1,2,3}\in(+,-,-), \\ -\left(C_F-\frac{C_A}{2}\right) & x_{1,2,3}\in(+,-,+), \\ 0 & x_{1,2,3}\in(-,-,+), \\ -\frac{C_A}{2} & x_{1,2,3}\in(-,+,+), \\ C_F-\frac{C_A}{2} & x_{1,2,3}\in(-,+,-), \\ 0 & x_{1,2,3}\in(+,+,-), \end{cases} + \mathcal{O}(a_s^2),$$